Exercises

Portfolio Optimization: Theory and Application
Chapter 10 – Portfolios with Alternative Risk Measures

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Exercise 10.1: Computing alternative measures of risk

Generate 10 000 samples following a normal distribution, plot the histogram, and compute the following measures:

- mean
- variance and standard deviation
- semi-variance and semi-deviation
- tail measures (VaR, CVaR, and EVaR) based on raw data
- tail measures (VaR, CVaR, and EVaR) based on a Gaussian approximation.

Exercise 10.2: CVaR in variational convex form

Consider the following expression for the CVaR:

$$\mathrm{CVaR}_\alpha = \mathbb{E}\left[\xi \mid \xi \geq \mathrm{VaR}_\alpha\right].$$

Show that it can be rewritten in a convex variational form as:

$$\mathrm{CVaR}_\alpha = \inf_{\tau}\left\{\tau + \frac{1}{1-\alpha}\mathbb{E}\left[(\xi-\tau)^+\right]\right\},$$

where the optimal τ precisely equals VaR_α.

Exercise 10.3: Sanity check for variational computation of CVaR

Generate 10 000 samples of the random variable ξ following a normal distribution and compute the CVaR as

$$\mathrm{CVaR}_\alpha = \mathbb{E}\left[\xi \mid \xi \geq \mathrm{VaR}_\alpha\right].$$

Verify numerically that the variational expression for the CVaR gives the same result:

$$\mathrm{CVaR}_\alpha = \inf_\tau \left\{\tau + \frac{1}{1-\alpha}\mathbb{E}\left[(\xi-\tau)^+\right]\right\}.$$

Exercise 10.4: CVaR vs. downside risk

Consider the following two measures of risk in terms of the loss random variable ξ:

- downside risk in the form of lower partial moment (LPM) with $\alpha = 1$:

$$\mathrm{LPM}_1 = \mathbb{E}\left[(\xi-\xi_0)^+\right];$$

- CVaR:

$$\mathrm{CVaR}_\alpha = \mathbb{E}\left[\xi \mid \xi \geq \mathrm{VaR}_\alpha\right].$$

Rewrite LPM_1 in the form of CVaR_α and the other way around. Hint: use $\xi_0 = \mathrm{VaR}_\alpha$.

Exercise 10.5: Log-sum-exp function as exponential cone

Show that the following convex constraint involving the perspective operator on the log-sum-exp function,

$$s \geq t\,\log\left(e^{x_1/t} + e^{x_2/t}\right),$$

for $t > 0$, can be rewritten in terms of the exponential cone $\mathcal{K}_{\mathrm{exp}}$ as

$$\begin{array}{c} t \geq u_1 + u_2, \\ (x_i - s, t, u_i) \in \mathcal{K}_{\mathrm{exp}}, \qquad i = 1,2, \end{array}$$

where

$$\mathcal{K}_{\mathrm{exp}} \triangleq \left\{(a,b,c) \mid c \geq b\,e^{a/b}, b > 0\right\} \cup \left\{(a,b,c) \mid a \leq 0, b = 0, c \geq 0\right\}.$$

Exercise 10.6: Drawdown and path-dependency

a. Generate 10 000 samples of returns following a normal distribution.
b. Compute and plot the cumulative returns, and plot the drawdown.
c. Randomly reorder the original returns and plot again.
d. Repeat a few times to observe the path-dependency property of the drawdown.

Exercise 10.7: Semi-variance portfolios

a. Download market data corresponding to N assets (e.g., stocks or cryptocurrencies) during a period with T observations, $\boldsymbol{r}_1, \dots, \boldsymbol{r}_T \in \mathbb{R}^N$.
b. Solve the minimization of the semi-variance in a nonparametric way (reformulate it as a quadratic program):

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{minimize}} & \frac{1}{T}\sum_{t=1}^{T}\left(\left(\tau - \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t\right)^+\right)^2 \\ \text{subject to} & \boldsymbol{w} \geq \boldsymbol{0}, \quad \boldsymbol{1}^\mathsf{T}\boldsymbol{w} = 1. \end{array}$$

c. Solve the parametric approximation based on the quadratic program:

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{minimize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{M}\boldsymbol{w} \\ \text{subject to} & \boldsymbol{w} \geq \boldsymbol{0}, \quad \boldsymbol{1}^\mathsf{T}\boldsymbol{w} = 1, \end{array}$$

where

$$\boldsymbol{M} = \mathbb{E}\left[\left(\tau\boldsymbol{1} - \boldsymbol{r}\right)^+ \left(\left(\tau\boldsymbol{1} - \boldsymbol{r}\right)^+\right)^\mathsf{T}\right].$$

d. Comment on the goodness of the approximation.

Exercise 10.8: CVaR portfolios

a. Download market data corresponding to N assets (e.g., stocks or cryptocurrencies) during a period with T observations, $\boldsymbol{r}_1, \dots, \boldsymbol{r}_T \in \mathbb{R}^N$.
b. Solve the minimum CVaR portfolio as the following linear program for different values of the parameter α:

$$\begin{array}{lll} \underset{\boldsymbol{w},\tau,\boldsymbol{u}}{\text{minimize}} & \tau + \frac{1}{1-\alpha}\frac{1}{T}\sum_{t=1}^{T} u_t & \\ \text{subject to} & 0 \leq u_t \geq -\boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t - \tau, & t = 1, \dots, T, \\ & \boldsymbol{w} \geq \boldsymbol{0}, \quad \boldsymbol{1}^\mathsf{T}\boldsymbol{w} = 1. & \end{array}$$

c. Observe how many observations are actually used ($u_t > 0$) for the different values of α.
d. Add some small perturbation or noise to the sequence of returns $\boldsymbol{r}_1, \dots, \boldsymbol{r}_T$ and repeat the experiment to observe the sensitivity of the solutions to data perturbation.

Exercise 10.9: Mean–Max-DD formulation as an LP

The mean–Max-DD formulation replaces the usual variance term $\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}$ by the Max-DD as a measure of risk, defined as

$$\text{Max-DD}(\boldsymbol{w}) = \max_{1\leq t\leq T} D_t(\boldsymbol{w}),$$

where $D_t(\boldsymbol{w})$ is the drawdown at time t. This leads to the problem formulation

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \lambda \max\limits_{1\leq t\leq T}\left\{\max\limits_{1\leq \tau\leq t} \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_\tau^{\text{cum}} - \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t^{\text{cum}}\right\} \\ \text{subject to} & \boldsymbol{w} \in \mathcal{W}. \end{array}$$

Show that it can be rewritten as the following problem ($u_0 \triangleq -\infty$):

$$\begin{array}{ll} \underset{\boldsymbol{w},\boldsymbol{u},s}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \lambda\, s \\ \text{subject to} & \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t^{\text{cum}} \le u_t \le s + \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t^{\text{cum}}, \quad t = 1,\dots,T, \\ & u_{t-1} \le u_t, \\ & \boldsymbol{w} \in \mathcal{W}, \end{array}$$

which is a linear program (assuming $\mathcal{W}$ only contains linear constraints).

Exercise 10.10: Mean–Ave-DD formulation as an LP

The mean–Ave-DD formulation replaces the usual variance term $\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}$ by the Ave-DD as a measure of risk, defined as

$$\text{Ave-DD} = \frac{1}{T}\sum_{1\le t\le T} D_t(\boldsymbol{w}),$$

where $D_t(\boldsymbol{w})$ is the drawdown at time t. This leads to the problem formulation

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \lambda \dfrac{1}{T}\displaystyle\sum_{t=1}^{T}\left(\max_{1\le\tau\le t} \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_\tau^{\text{cum}} - \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t^{\text{cum}}\right) \\ \text{subject to} & \boldsymbol{w} \in \mathcal{W}. \end{array}$$

Show that it can be rewritten as the following problem ($u_0 \triangleq -\infty$):

$$\begin{array}{ll} \underset{\boldsymbol{w},\boldsymbol{u},s}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \lambda\, s \\ \text{subject to} & \frac{1}{T}\sum_{t=1}^{T} u_t \le \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t^{\text{cum}} + s, \\ & \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t^{\text{cum}} \le u_t, \qquad t = 1,\dots,T, \\ & u_{t-1} \le u_t, \\ & \boldsymbol{w} \in \mathcal{W}, \end{array}$$

which is a linear program (assuming $\mathcal{W}$ only contains linear constraints).

Exercise 10.11: Mean–CVaR-DD formulation as an LP

The mean–CVaR-DD formulation replaces the usual variance term $\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}$ by the CVaR-DD as a measure of risk, expressed in a variational form as

$$\text{CVaR-DD}(\boldsymbol{w}) = \inf_{\tau}\left\{\tau + \frac{1}{1-\alpha}\frac{1}{T}\sum_{t=1}^{T}(D_t(\boldsymbol{w}) - \tau)^+\right\},$$

where $D_t(\boldsymbol{w})$ is the drawdown at time t. This leads to the problem formulation

$$\begin{array}{ll} \underset{\boldsymbol{w},\tau}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \lambda\left(\tau + \dfrac{1}{1-\alpha}\dfrac{1}{T}\displaystyle\sum_{t=1}^{T}\left(\max_{1\le\tau'\le t} \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_{\tau'}^{\text{cum}} - \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t^{\text{cum}} - \tau\right)^+\right) \\ \text{subject to} & \boldsymbol{w} \in \mathcal{W}. \end{array}$$

Show that it can be rewritten as the following problem ($u_0 \triangleq -\infty$):

$$
\begin{array}{ll}
\underset{\boldsymbol{w},\tau,s,\boldsymbol{z},\boldsymbol{u}}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \lambda\, s\\
\text{subject to} & s \geq \tau + \frac{1}{1-\alpha}\frac{1}{T}\sum_{t=1}^{T} z_t,\\
& 0 \leq z_t \geq u_t - \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t^{\text{cum}} - \tau, \qquad t = 1,\dots,T,\\
& \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t^{\text{cum}} \leq u_t,\\
& u_{t-1} \leq u_t,\\
& \boldsymbol{w} \in \mathcal{W},
\end{array}
$$

which is a linear program (assuming $\mathcal{W}$ only contains linear constraints).

Exercise 10.12: Mean–EVaR-DD formulation as a convex problem

The mean–EVaR-DD formulation replaces the usual variance term $\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}$ by the EVaR-DD as a measure of risk, defined as

$$
\text{EVaR-DD}(\boldsymbol{w}) = \inf_{z>0}\left\{z^{-1}\log\left(\frac{1}{1-\alpha}\frac{1}{T}\sum_{t=1}^{T}\exp(zD_t(\boldsymbol{w}))\right)\right\},
$$

where $D_t(\boldsymbol{w})$ is the drawdown at time t defined as

$$
D_t(\boldsymbol{w}) = \max_{1\leq\tau\leq t} \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_\tau^{\text{cum}} - \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t^{\text{cum}}.
$$

a. Write down the mean–EVaR-DD portfolio formulation in convex form.
b. Further rewrite the problem in terms of the exponential cone.