Exercises

Portfolio Optimization: Theory and Application
Chapter 12 – Graph-Based Portfolios

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Exercise 12.1: Learning heavy-tailed financial graphs

a. Download market data corresponding to N assets (e.g., stocks or cryptocurrencies) during a period with T observations, and form the data matrix $\boldsymbol{X} \in \mathbb{R}^{T \times N}$.

b. Learn a Gaussian MRF graph:

$$\begin{array}{ll} \underset{\boldsymbol{w} \geq \boldsymbol{0}}{\text{maximize}} & \log \text{gdet}(\mathcal{L}(\boldsymbol{w})) - \text{Tr}(\mathcal{L}(\boldsymbol{w})\boldsymbol{S}) \\ \text{subject to} & \mathfrak{d}(\boldsymbol{w}) = \boldsymbol{1}, \end{array}$$

where $\boldsymbol{S}$ is the sample covariance matrix of the data, $\mathcal{L}(\boldsymbol{w})$ is the Laplacian operator that produces the Laplacian matrix $\boldsymbol{L}$ from the weights $\boldsymbol{w}$, and $\mathfrak{d}(\boldsymbol{w})$ is the degree operator that gives the degrees of the nodes.

c. Learn a heavy-tailed MRF graph directly:

$$\begin{array}{ll} \underset{\boldsymbol{w} \geq \boldsymbol{0}}{\text{maximize}} & \log \text{gdet}(\mathcal{L}(\boldsymbol{w})) - \dfrac{N+\nu}{T} \displaystyle\sum_{t=1}^{T} \log \left(\nu + (\boldsymbol{x}^{(t)})^\mathsf{T} \mathcal{L}(\boldsymbol{w}) \boldsymbol{x}^{(t)}\right) \\ \text{subject to} & \mathfrak{d}(\boldsymbol{w}) = \boldsymbol{1}, \end{array}$$

where $\boldsymbol{x}^{(t)}$ is the tth row of the data matrix $\boldsymbol{X}$.

d. Learn a heavy-tailed MRF graph by solving the sequence of Gaussianized problems for $k = 1, 2, \dots$

$$\begin{array}{ll} \underset{\boldsymbol{w} \geq \boldsymbol{0}}{\text{maximize}} & \log \text{gdet}(\mathcal{L}(\boldsymbol{w})) - \text{Tr}(\mathcal{L}(\boldsymbol{w})\boldsymbol{S}^k) \\ \text{subject to} & \mathfrak{d}(\boldsymbol{w}) = \boldsymbol{1}, \end{array}$$

where $\boldsymbol{S}^k$ is the weighted sample covariance matrix

$$\boldsymbol{S}^k = \frac{1}{T}\sum_{t=1}^{T} w_t^k \times \boldsymbol{x}^{(t)}(\boldsymbol{x}^{(t)})^\mathsf{T},$$

with weights $w_t^k = \dfrac{N+\nu}{\nu + (\boldsymbol{x}^{(t)})^\mathsf{T}\mathcal{L}(\boldsymbol{w}^k)\boldsymbol{x}^{(t)}}$.

e. Plot the graphs and compare them visually.

f. Compute the dendrogram for each of the graphs and compare them.

Exercise 12.2: Hierarchical $1/N$ portfolio

a. Download market data corresponding to N assets during a period with T observations, and form the data matrix $\boldsymbol{X} \in \mathbb{R}^{T\times N}$.
b. Learn the graph distance matrix based on (i) a simple distance matrix from the distance between the time series of asset pairs, and (ii) a heavy-tailed MRF graph formulation.
c. Construct the hierarchical $1/N$ portfolio.
d. Plot the portfolio allocation and perform a backtest (comparing with the $1/N$ portfolio).

Exercise 12.3: Hierarchical risk parity (HRP) portfolio

a. Download market data corresponding to N assets during a period with T observations, and form the data matrix $\boldsymbol{X} \in \mathbb{R}^{T\times N}$.
b. Learn the graph distance matrix based on (i) a simple distance matrix from the distance between the time series of asset pairs, and (ii) a heavy-tailed MRF graph formulation.
c. Construct the HRP portfolio.
d. Plot the portfolio allocation and perform a backtest (comparing with the inverse-variance portfolio).

Exercise 12.4: Hierarchical equal risk contribution (HERC) portfolio

a. Download market data corresponding to N assets during a period with T observations, and form the data matrix $\boldsymbol{X} \in \mathbb{R}^{T\times N}$.
b. Learn the graph distance matrix based on (i) a simple distance matrix from the distance between the time series of asset pairs, and (ii) a heavy-tailed MRF graph formulation.
c. Construct the HERC portfolio.
d. Plot the portfolio allocation and perform a backtest (comparing with the hierarchical $1/N$ portfolio and HRP portfolio).

Exercise 12.5: From minimum variance portfolio to hierarchical portfolio

a. Derive the inverse of the 2×2 block matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_A & \boldsymbol{\Sigma}_{AB} \\ \boldsymbol{\Sigma}_{BA} & \boldsymbol{\Sigma}_B \end{bmatrix},$$

identifying the Schur complements of $\boldsymbol{\Sigma}_A$ and $\boldsymbol{\Sigma}_B$ defined, respectively, as

$$\begin{aligned} \boldsymbol{\Sigma}_A^{\mathrm{c}} &= \boldsymbol{\Sigma}_A - \boldsymbol{\Sigma}_{AB}\boldsymbol{\Sigma}_B^{-1}\boldsymbol{\Sigma}_{BA}, \\ \boldsymbol{\Sigma}_B^{\mathrm{c}} &= \boldsymbol{\Sigma}_B - \boldsymbol{\Sigma}_{BA}\boldsymbol{\Sigma}_A^{-1}\boldsymbol{\Sigma}_{AB}. \end{aligned}$$

b. Derive the global minimum variance portfolio

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{minimize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w} \\ \text{subject to} & \boldsymbol{w}^\mathsf{T}\mathbf{1} = 1 \end{array}$$

in the form (up to a scaling factor)

$$\boldsymbol{w} \propto \boldsymbol{\Sigma}^{-1}\mathbf{1} = \begin{bmatrix} (\boldsymbol{\Sigma}_A^{\mathrm{c}})^{-1}\boldsymbol{b}_A \\ (\boldsymbol{\Sigma}_B^{\mathrm{c}})^{-1}\boldsymbol{b}_B \end{bmatrix},$$

where

$$\begin{aligned} \boldsymbol{b}_A &= \mathbf{1} - \boldsymbol{\Sigma}_{AB}\boldsymbol{\Sigma}_B^{-1}\mathbf{1}, \\ \boldsymbol{b}_B &= \mathbf{1} - \boldsymbol{\Sigma}_{BA}\boldsymbol{\Sigma}_A^{-1}\mathbf{1}. \end{aligned}$$

c. Rewrite the solution in the form

$$\boldsymbol{w} \propto \begin{bmatrix} \frac{1}{\nu\left(\boldsymbol{\Sigma}_A^{\mathrm{c}}, \boldsymbol{b}_A\right)} \boldsymbol{w}\left(\boldsymbol{\Sigma}_A^{\mathrm{c}}, \boldsymbol{b}_A\right) \\ \frac{1}{\nu\left(\boldsymbol{\Sigma}_B^{\mathrm{c}}, \boldsymbol{b}_B\right)} \boldsymbol{w}\left(\boldsymbol{\Sigma}_B^{\mathrm{c}}, \boldsymbol{b}_B\right) \end{bmatrix},$$

for properly defined (normalized) allocation $\boldsymbol{w}(\boldsymbol{\Sigma}, \boldsymbol{b})$ and measure of risk $\nu(\boldsymbol{\Sigma}, \boldsymbol{b})$.