Portfolio Optimization

Portfolios with Alternative Risk Measures

Daniel P. Palomar (2024). *Portfolio Optimization: Theory and Application.*
Cambridge University Press.

portfoliooptimizationbook.com

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Outline

Abstract

Markowitz's mean-variance portfolio optimizes the trade-off between expected return and risk, measured by variance, which, despite being intuitive, is a simplistic measure that penalizes both unwanted losses and desired gains while ignoring the shape of the returns' distribution function. Rather than focusing on the middle part of the distribution, as volatility does, the tail of the distribution characterizes the big losses. These slides explore various alternative and more sophisticated risk measures proposed over the past seven decades, such as downside risk, semivariance, value-at-risk, conditional value-at-risk, expected shortfall, and drawdown, and, more importantly, how to incorporate these measures into the portfolio formulation in a manageable way (Palomar 2024, chap. 10).

Outline

Introduction

- **Markowitz's mean-variance portfolio:**
  - Balances expected return and risk (variance).
  - Optimization problem:

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \frac{\lambda}{2}\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w} \\ \text{subject to} & \boldsymbol{w} \in \mathcal{W} \end{array}$$

  where
    - λ: risk-aversion hyper-parameter
    - $\mathcal{W}$: constraint set, e.g., $\mathcal{W} = \{\boldsymbol{w} \mid \boldsymbol{1}^\mathsf{T}\boldsymbol{w} = 1, \boldsymbol{w} \geq \boldsymbol{0}\}$.
- **Limitations of variance as a risk measure:**
  - Variance ($\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}$) or volatility ($\sqrt{\boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}}$) may not predict out-of-sample performance well.
  - Markowitz highlighted these limitations as early as 1959 (Markowitz 1959).
- **Alternative risk measures:**
  - Academics and practitioners have sought other risk measures beyond variance.
  - Coherent risk measures are a notable category, introduced for their desirable properties.

Exploring alternative risk measures

- These slides explores various alternative risk measures, including:
  - Downside risk
  - Semivariance
  - Semi-deviation
  - Value-at-Risk (VaR)
  - Conditional Value-at-Risk (CVaR)
  - Expected Shortfall (ES)
  - Drawdown

- The focus is on incorporating these measures into portfolio formulation.

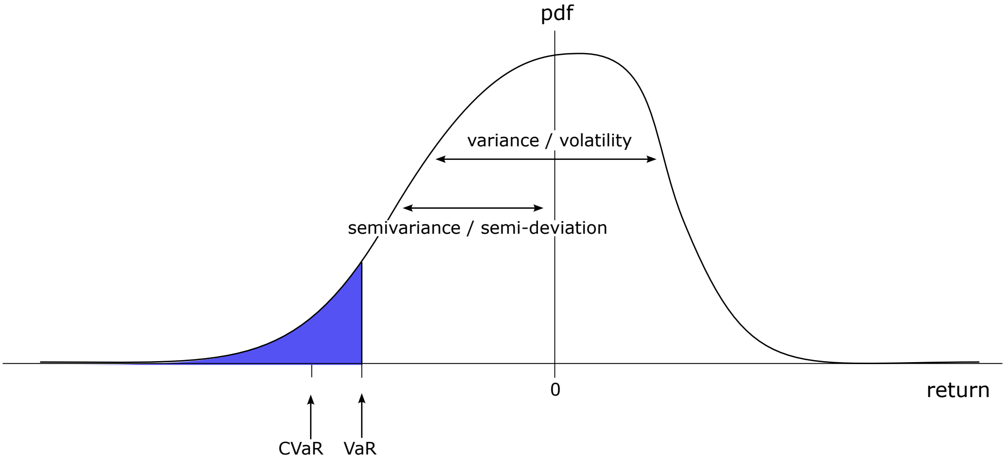
Outline

Alternative risk measures

- **Portfolio return as a random variable:**
  - Portfolio return R:
  $$R = \boldsymbol{w}^{\mathsf{T}}\boldsymbol{r}$$
  - $\boldsymbol{r}$: vector of random returns for N assets.
  - Return R characterized by its probability distribution function (pdf).
- **Condensing information from the pdf:**
  - pdf information typically reduced to key numbers: mean (expected return) and standard deviation (risk).
  - Choice of risk measure has been researched since the 1950s.
- **Alternative risk measures:**
  - Search for risk measures with desirable properties.
  - Popular risk measures that reduce the pdf to a number: semivariance/semi-deviation, VaR, and CVaR.
  - Drawdown is a special case as it is not invariant to the order of returns.

Alternative risk measures

Illustration of return distribution and measures of risk:



Downside risk

- **Investor attitudes towards losses and gains:**
  - Investors perceive downside losses differently from upside gains.
  - Markowitz suggested semivariance as a more appropriate risk measure than variance.

- **Downside risk:**
  - Refers to measures quantifying losses below a certain threshold.
  - Considered more meaningful than symmetric measures like variance or volatility.
  - Effectiveness depends on the asymmetry of return distributions.

- **Semivariance and semi-deviation:**
  - **Semivariance (SV):**

$$\mathsf{SV} = \mathbb{E}\left[\left((\mu - R)^{+}\right)^{2}\right]$$

    - Only accounts for returns below the mean.
  - **Semi-deviation:** Square root of semivariance.
  - **Sortino ratio:** Expected return to semi-deviation ratio, analogous to Sharpe ratio.

Downside risk

- **Lower partial moment (LPM):** Generalizes semivariance:

$$\mathsf{LPM}_\alpha = \mathbb{E}\left[\left((\tau - R)^+\right)^\alpha\right]$$

  where
  - τ: disaster level (minimum acceptable return).
  - α: reflects investor's attitude towards falling short of τ
    - $\alpha > 1$ naturally fits a risk-averse investor
    - $\alpha = 1$ corresponds to a neutral investor
    - $0 < \alpha < 1$ is suitable for risk-seeking behavior.
  - By adjusting α and τ, various downside measures can be derived, including semivariance.

- **Mean-risk trade-off visualization:**
  - Traditional portfolio theory plots mean vs. volatility.
  - Similarly, mean vs. risk ($\mathsf{LPM}_\alpha^{1/\alpha}$) can be plotted for downside risk measures.

Tail measures: VaR, CVaR, and EVaR

- **Tail measures overview:**
  - Focus on the tail of the distribution representing significant losses.
  - Defined in terms of loss $\xi = -\boldsymbol{w}^\mathsf{T}\boldsymbol{r}$.
  - Contrast with variance/volatility and semivariance/semi-deviation which measure dispersion.

- **Value-at-risk (VaR):**
  - Maximum loss at a specified confidence level.
  - Defined as:

$$\mathrm{VaR}_\alpha = \inf\left\{\xi_0 : \Pr\left[\xi \leq \xi_0\right] \geq \alpha\right\}$$

  - Nonconvex.

- **Conditional value-at-risk (CVaR):**
  - Average of losses exceeding VaR.
  - Defined as:

$$\mathrm{CVaR}_\alpha = \mathbb{E}\left[\xi \mid \xi \geq \mathrm{VaR}_\alpha\right]$$

Tail measures: VaR, CVaR, and EVaR*

- **Entropic value-at-risk (EVaR):**
  - Tightest upper bound from Chernoff inequality for VaR.
  - Defined as:
$$\mathrm{EVaR}_\alpha = \inf_{z>0} \left\{ z^{-1} \log \left(\frac{1}{1-\alpha} \mathbb{E}\left[\exp(z\xi)\right] \right) \right\}$$

- **Connections among tail measures:**
  - Monotonicity:
$$\mathrm{VaR}_\alpha \leq \mathrm{CVaR}_\alpha \leq \mathrm{EVaR}_\alpha$$
  - "Average VaR" expression:
$$\mathrm{CVaR}_\alpha = \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{VaR}_u \, \mathrm{d}u$$
  - Limiting behavior: VaR, CVaR, and EVaR converge to the maximal value of pdf support as $\alpha \to 1$.

Tail measures: VaR, CVaR, and EVaR

- **Tail measures under the Gaussian distribution:**

$$
\begin{aligned}
\mathrm{VaR}_\alpha &= \mu + \sigma\,\Phi^{-1}(\alpha)\\
\mathrm{CVaR}_\alpha &= \mu + \sigma\frac{\phi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}\\
\mathrm{EVaR}_\alpha &= \mu + \sigma\sqrt{-2\mathrm{log}(1-\alpha)},
\end{aligned}
$$

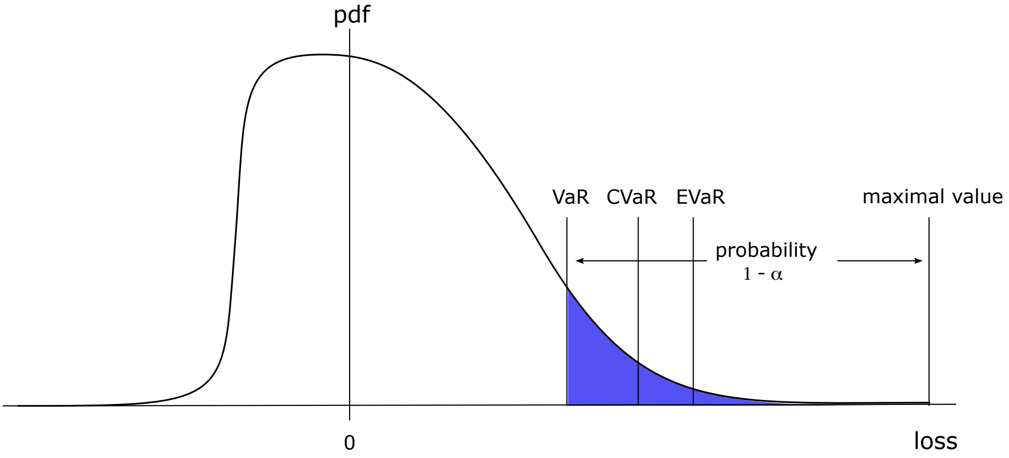
  where:

  - μ and σ are the mean and standard deviation,
  - Φ denotes the standard normal distribution function,
  - ϕ its density function, and
  - $\Phi^{-1}(\alpha)$ the α-quantile of Φ.

- Minimizing these tail measures under the Gaussian distribution amounts to simply minimizing the standard deviation σ or,equivalently, the variance σ^2.

Tail measures: VaR, CVaR, and EVaR

Illustration of loss distribution and tail measures (VaR, CVaR, and EVaR) (as well as the maximal value) in the context of the pdf of the loss:



Convex characterization of CVaR

- CVaR computation requires the VaR:

$$\mathrm{CVaR}_\alpha = \frac{1}{1-\alpha}\mathbb{E}\left[\xi \times I\{\xi \geq \mathrm{VaR}_\alpha\}\right] = \mathrm{VaR}_\alpha + \frac{1}{1-\alpha}\mathbb{E}\left[(\xi - \mathrm{VaR}_\alpha)^+\right]$$

- **Variational form of CVaR:**

$$\mathrm{CVaR}_\alpha = \inf_\tau \left\{\tau + \frac{1}{1-\alpha}\mathbb{E}\left[(\xi - \tau)^+\right]\right\}$$

  - No prior VaR requires: optimal τ equals VaR.

- **Portfolio optimization context:**

$$\mathrm{CVaR}_\alpha(\boldsymbol{w}) = \inf_\tau F_\alpha(\boldsymbol{w}, \tau)$$

where $F_\alpha(\boldsymbol{w}, \tau)$ is a **convex** auxiliary function:

$$F_\alpha(\boldsymbol{w}, \tau) = \tau + \frac{1}{1-\alpha}\mathbb{E}\left[(-\boldsymbol{w}^\mathsf{T}\boldsymbol{r} - \tau)^+\right].$$

From downside risk to CVaR*

- **CVaR and downside risk relationship:**
  - CVaR is related to downside risk (LPM) with $\alpha = 1$:

$$\mathsf{LPM}_1 = \mathbb{E}\left[(\tau - R) \times I\{R \leq \tau\}\right] = \mathbb{E}\left[(\xi - (-\tau)) \times I\{\xi \geq -\tau\}\right]$$

  - Loss $\xi = -R$.
- **Disaster level and VaR:**
  - Setting disaster level $\tau = -\mathsf{VaR}_\alpha$:

$$\mathsf{LPM}_1 = \mathbb{E}\left[(\xi - \mathsf{VaR}_\alpha) \times I\{\xi \geq \mathsf{VaR}_\alpha\}\right] = (1 - \alpha)\mathbb{E}\left[(\xi - \mathsf{VaR}_\alpha) \mid \xi \geq \mathsf{VaR}_\alpha\right]$$

- **Comparison with CVaR:**
  - CVaR definition:

$$\mathsf{CVaR}_\alpha = \mathbb{E}\left[\xi \mid \xi \geq \mathsf{VaR}_\alpha\right]$$

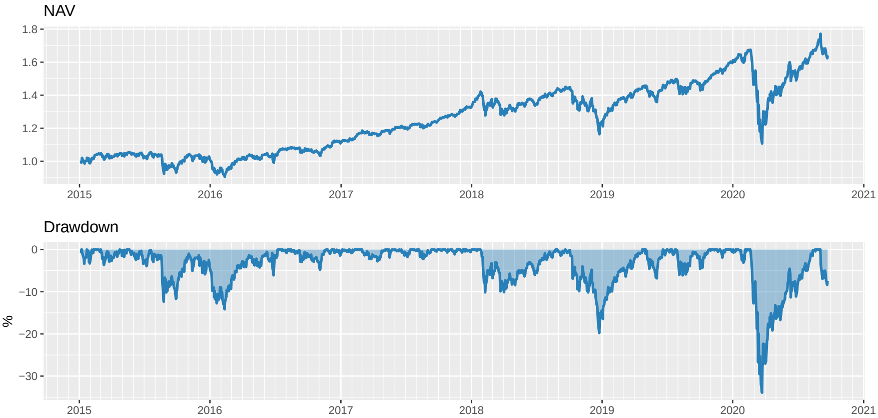
  - LPM_1 measures expected excess loss above VaR_α after shifting to the origin.
  - CVaR measures expected loss above VaR_α.
- **Key difference:**
  - VaR level is determined within the CVaR calculation.
  - Downside risk requires a predefined disaster level.

Drawdown

- **Drawdown concept:**
  - Measures investor's "suffering" from monitoring cumulative return or wealth.
  - Focuses solely on downside events, ignoring upside movements.
  - Loss measured relative to past maximum, reflecting human psychology.
- **High watermark (HWM):**
  - Historical peak of value $X(t)$ up to time t:
$$\text{HWM}(t) = \max_{1 \leq \tau \leq t} X(\tau).$$
- **Drawdown definitions:**
  - **Absolute drawdown:**
$$D(t) = \text{HWM}(t) - X(t);$$
  - **Normalized drawdown:**
$$\bar{D}(t) = \frac{\text{HWM}(t) - X(t)}{\text{HWM}(t)}.$$

Drawdown

Illustration of net asset value (NAV) curve of S&P 500 and corresponding normalized drawdown:

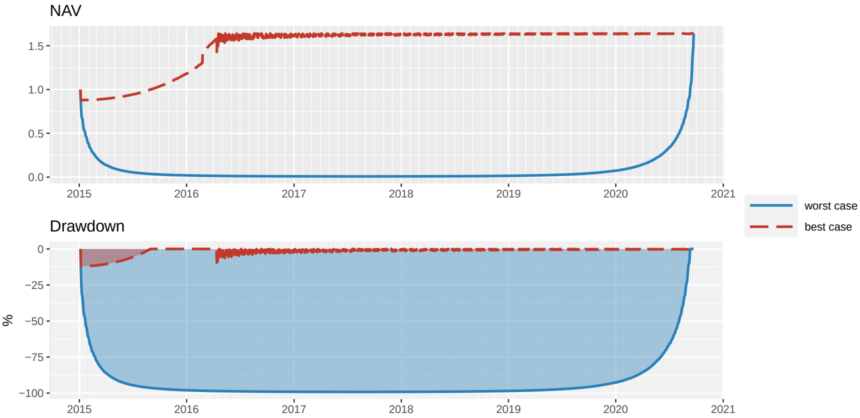


Drawdown: Path-dependency

- **Drawdown as a path-dependent measure:**
  - Drawdown is sensitive to the sequence of returns.
  - Contrasts with other risk measures that are indifferent to return order.
- **Impact of return order:**
  - Next figure illustrates the effect of return order on drawdown.
  - Best case scenario: drawdown peaks at approximately 12.5%.
  - Worst case scenario: drawdown approaches 100%.
  - Original sequence drawdown was 34%.

Drawdown: Path-dependency

Effect of ordering of returns in the cumulative return and drawdown:



Drawdown: Single-number summarization

To condense the drawdown curve into a single metric, various methods can be used:

- **Maximum drawdown (Max-DD):**
  - Represents the largest single drop from peak to trough over the period:

$$\text{Max-DD} = \max_{1 \leq t \leq T} D(t)$$

- **Average drawdown (Ave-DD):**
  - Calculates the mean of all drawdowns over the period:

$$\text{Ave-DD} = \frac{1}{T} \sum_{1 \leq t \leq T} D(t)$$

- **CVaR of drawdown (CVaR-DD) or conditional drawdown at risk (CDaR):**
  - Measures the expected drawdown in the worst α% of cases:

$$\text{CDaR}_\alpha = \mathbb{E}\left[D(t) \mid D(t) \geq \text{VaR}_\alpha\right]$$

  - VaR_α is the value at risk of drawdown $D(t)$ at confidence level α.

Outline

Outline

Downside risk portfolios

- **Objective with downside risk:**

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \lambda\,\mathbb{E}\left[\left((\tau - \boldsymbol{w}^\mathsf{T}\boldsymbol{r})^+\right)^\alpha\right] \\ \text{subject to} & \boldsymbol{w} \in \mathcal{W}. \end{array}$$

- **Approximating expectation with sample mean:**

$$\mathbb{E}\left[\left((\tau - \boldsymbol{w}^\mathsf{T}\boldsymbol{r})^+\right)^\alpha\right] \approx \frac{1}{T}\sum_{t=1}^{T}\left((\tau - \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t)^+\right)^\alpha.$$

- **Similar to the variance approximation:**

$$\mathbb{E}\left[\left(\boldsymbol{w}^\mathsf{T}(\boldsymbol{r} - \boldsymbol{\mu})\right)^2\right] \approx \frac{1}{T}\sum_{t=1}^{T}\left[\left(\boldsymbol{w}^\mathsf{T}(\boldsymbol{r} - \boldsymbol{\mu})\right)^2\right] = \boldsymbol{w}^\mathsf{T}\hat{\boldsymbol{\Sigma}}\boldsymbol{w},$$

where $\hat{\boldsymbol{\Sigma}}$ is the sample covariance matrix.

Mean-downside risk portfolio formulation

- **Final mean-downside risk formulation:** includes all return observations

$$\begin{array}{ll}\underset{\mathbf{w}}{\text{maximize}} & \mathbf{w}^{\mathsf{T}}\boldsymbol{\mu} - \lambda\frac{1}{T}\sum_{t=1}^{T}\left(\left(\tau - \mathbf{w}^{\mathsf{T}}\mathbf{r}_t\right)^{+}\right)^{\alpha}\\ \text{subject to} & \mathbf{w} \in \mathcal{W}\end{array}$$

- **Optimization without nondifferentiable operator $(\cdot)^{+}$:**

$$\begin{array}{ll}\underset{\mathbf{w},\{s_t\}}{\text{maximize}} & \mathbf{w}^{\mathsf{T}}\boldsymbol{\mu} - \lambda\frac{1}{T}\sum_{t=1}^{T} s_t^{\alpha}\\ \text{subject to} & 0 \leq s_t \geq \tau - \mathbf{w}^{\mathsf{T}}\mathbf{r}_t, \quad t = 1, \dots, T\\ & \mathbf{w} \in \mathcal{W}.\end{array}$$

- **Convex problem** for common choices of α:
  - Linear program for $\alpha = 1$.
  - Quadratic program for $\alpha = 2$ (semivariance portfolio).
  - General convex program for $\alpha = 3$.

Semivariance portfolios

- **Variance and covariance matrix:**
  - Variance expressed via covariance matrix $\boldsymbol{\Sigma}$:
$$\mathbb{E}\left[\left(\boldsymbol{w}^\mathsf{T}(\boldsymbol{r}-\boldsymbol{\mu})\right)^2\right] = \boldsymbol{w}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{w}$$
  - Covariance matrix definition:
$$\boldsymbol{\Sigma} = \mathbb{E}\left[(\boldsymbol{r}-\boldsymbol{\mu})(\boldsymbol{r}-\boldsymbol{\mu})^\mathsf{T}\right]$$
- **Semivariance approximation:**
  - Seeking a similar expression for semivariance:
$$\mathbb{E}\left[\left((\tau-\boldsymbol{w}^\mathsf{T}\boldsymbol{r})^+\right)^2\right] \approx \boldsymbol{w}^\mathsf{T}\boldsymbol{M}\boldsymbol{w}$$
  - No exact exogenous matrix $\boldsymbol{M}$ (independent of $\boldsymbol{w}$) for semivariance.
- **Markowitz's semivariance matrix:**
  - Exact semivariance matrix proposed by Markowitz:
$$\boldsymbol{M}(\boldsymbol{w}) = \mathbb{E}\left[(\tau\mathbf{1}-\boldsymbol{r})(\tau\mathbf{1}-\boldsymbol{r})^\mathsf{T} \times I\{\tau > \boldsymbol{w}^\mathsf{T}\boldsymbol{r}\}\right]$$
  - Matrix is endogenous, depending on portfolio $\boldsymbol{w}$, and not suitable for optimization.

Semivariance Portfolios

- **Heuristic approximations:**
  - Proposed approximation:

$$\boldsymbol{M} = \mathbb{E}\left[(\tau\mathbf{1} - \boldsymbol{r})^{+}\left((\tau\mathbf{1} - \boldsymbol{r})^{+}\right)^{\mathsf{T}}\right]$$

  - Other practical approaches have been explored.

- **Mean-semivariance formulation:**
  - By setting $\alpha = 2$ in mean–downside risk formulation:

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \boldsymbol{w}^{\mathsf{T}}\boldsymbol{\mu} - \frac{\lambda}{2}\boldsymbol{w}^{\mathsf{T}}\boldsymbol{M}\boldsymbol{w} \\ \text{subject to} & \boldsymbol{w} \in \mathcal{W} \end{array}$$

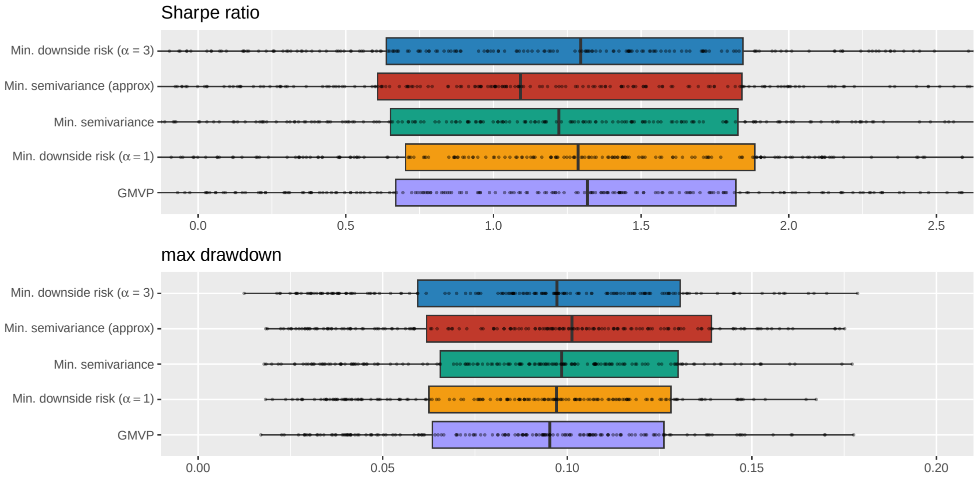
  - Offers a convenient approximation similar to mean–variance formulation.

Numerical experiments

- **Objective:**
  - Compare downside risk portfolios for different values of α (1, 2, and 3) based on the mean–downside risk formulation.
  - Exclude expected return from optimization to solely focus on risk measure impact.
  - Use the Global Minimum Variance Portfolio (GMVP) as a benchmark.
- **Setup:**
  - Analyze portfolios for $\alpha = 1$ (linear program), $\alpha = 2$ (quadratic program, with and without approximation), and $\alpha = 3$ (general convex program).
  - Evaluate over 200 realizations of 50 randomly selected stocks from the S&P 500 during 2015-2020.
  - Reoptimize portfolios monthly with a one-year lookback period.
- **Metrics for comparison:**
  - Sharpe ratio: Measures risk-adjusted return.
  - Maximum drawdown: Indicates the largest single drop from peak to trough during the observation period.

Numerical experiments

Backtest performance of different downside risk portfolios:



Outline

Naive attempt to CVaR portfolio

- Recall that
$$\text{CVaR}_\alpha = \mathbb{E}\left[\xi \mid \xi \geq \text{VaR}_\alpha\right].$$
- In practice, given T observations:
  - We can first order them in decreasing order:
  $$\xi_{[1]} \geq \xi_{[2]} \geq \xi_{[3]} \geq \cdots \geq \xi_{[T]}$$
  where $\xi_{[i]}$ denotes the i-th largest value.
  - Then, we can compute the empirical CVaR as the sample mean of the first αT samples:
  $$\text{CVaR}_\alpha \approx \frac{1}{\alpha T}\sum_{i=1}^{\alpha T} \xi_{[i]}.$$
- However, in terms of portfolio optimization, the loss depends on the portfolio $\boldsymbol{w}$ as $\xi_t = -\boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t$ and then the order of the loss values will change with $\boldsymbol{w}$!!:
$$(-\boldsymbol{w}^\mathsf{T}\boldsymbol{r})_{[1]} \geq (-\boldsymbol{w}^\mathsf{T}\boldsymbol{r})_{[2]} \geq \cdots \geq (-\boldsymbol{w}^\mathsf{T}\boldsymbol{r})_{[T]}.$$

Formulation for the CVaR portfolio

- **Mean-CVaR formulation:** Replaces variance with CVaR as risk measure:

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \lambda\,\mathrm{CVaR}_\alpha(\boldsymbol{w}) \\ \text{subject to} & \boldsymbol{w} \in \mathcal{W} \end{array}$$

- **Convex representation of CVaR:**

$$\mathrm{CVaR}_\alpha(\boldsymbol{w}) = \inf_\tau \left\{\tau + \frac{1}{1-\alpha}\mathbb{E}\left[(-\boldsymbol{w}^\mathsf{T}\boldsymbol{r} - \tau)^+\right]\right\}.$$

- **Convex mean-CVaR formulation:** Incorporates auxiliary variable τ into optimization:

$$\begin{array}{ll} \underset{\boldsymbol{w},\tau}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \lambda\left(\tau + \frac{1}{1-\alpha}\mathbb{E}\left[(-\boldsymbol{w}^\mathsf{T}\boldsymbol{r} - \tau)^+\right]\right) \\ \text{subject to} & \boldsymbol{w} \in \mathcal{W} \end{array}$$

Formulation for the CVaR portfolio

- **Approximation with sample mean:**
  - Approximates expectation with sample mean:

$$\mathbb{E}\left[(-\boldsymbol{w}^{\mathsf{T}}\boldsymbol{r}-\tau)^{+}\right] \approx \frac{1}{T}\sum_{t=1}^{T}(-\boldsymbol{w}^{\mathsf{T}}\boldsymbol{r}_t-\tau)^{+}.$$

- **Final formulation with auxiliary variables:**
  - Introduces T auxiliary variables $\boldsymbol{u}$ for linear programming:

$$\begin{array}{ll}\underset{\boldsymbol{w},\tau,\boldsymbol{u}}{\text{maximize}} & \boldsymbol{w}^{\mathsf{T}}\boldsymbol{\mu}-\lambda\left(\tau+\frac{1}{1-\alpha}\frac{1}{T}\sum_{t=1}^{T}u_t\right)\\ \text{subject to} & 0 \leq u_t \geq -\boldsymbol{w}^{\mathsf{T}}\boldsymbol{r}_t-\tau, \quad t=1,\dots,T\\ & \boldsymbol{w}\in\mathcal{W}\end{array}$$

- **Challenges with CVaR estimation:**
  - Tail events occur with low probability, leading to few samples characterizing the tail.
  - Numerical instability may arise, especially with high dimensions or insufficient samples.
  - Alternative methods include parametric distribution assumptions, worst-case CVaR characterizations, and extreme value theory applications.

Formulation for the EVaR portfolio*

- **Mean-EVaR formulation:**
  - Replaces variance with EVaR as a risk measure:

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \lambda\,\mathrm{EVaR}_\alpha(\boldsymbol{w}) \\ \text{subject to} & \boldsymbol{w} \in \mathcal{W} \end{array}$$

- **Convex representation of EVaR:**
  - Utilizes a change of variable for EVaR formulation:

$$\begin{array}{ll} \underset{\boldsymbol{w},t>0}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \lambda\left(t\,\log\left(\frac{1}{1-\alpha}\mathbb{E}\left[\exp(-t^{-1}\boldsymbol{w}^\mathsf{T}\boldsymbol{r})\right]\right)\right) \\ \text{subject to} & \boldsymbol{w} \in \mathcal{W} \end{array}$$

Formulation for the EVaR portfolio*

- **Sample mean approximation:**
  - Approximates expectation with sample mean:

$$\begin{array}{ll} \underset{\boldsymbol{w},t>0}{\text{maximize}} & \boldsymbol{w}^{\mathsf{T}}\boldsymbol{\mu} - \lambda\left(t\,\log\left(\sum_{t'=1}^{T}\left[\exp(-t^{-1}\boldsymbol{w}^{\mathsf{T}}\boldsymbol{r}_{t'})\right]\right) - t\,\log\left((1-\alpha)T\right)\right) \\ \text{subject to} & \boldsymbol{w} \in \mathcal{W} \end{array}$$

- **Solving the problem:** This problem can be solved in practice in a variety of ways:
  - via a general-purpose solver (since the problem is convex, it will find an optimal solution);
  - via a tailored interior-point method for convex problems (Ahmadi-Javid and Fallah-Tafti 2019);
  - via a convex modeling framework that can recognize the convexity of the log-sum-exp function and then performing bisection over t;
  - via a convex modeling framework that can recognize both the convexity of the log-sum-exp function and the convexity-preserving property of the perspective operator;
  - via a convex reformulation in terms of the *exponential cone* $\mathcal{K}_{\text{exp}}$ (Chares 2007), which some solvers and modeling frameworks can recognize (Cajas 2021)

Formulation for the worst-case portfolio

- **Worst-case risk focus:**
  - As $\alpha \to 1$, VaR, CVaR, and EVaR converge to the maximal loss value.
  - Emphasizes the worst possible return or loss scenario.

- **Worst-case risk formulation:**
  - Maximizes expected return while penalizing the worst loss:

$$
\begin{array}{ll}
\underset{\mathbf{w}}{\text{maximize}} & \mathbf{w}^\mathsf{T}\boldsymbol{\mu} - \lambda \max_{1\le t\le T}\{-\mathbf{w}^\mathsf{T}\mathbf{r}_t\}\\
\text{subject to} & \mathbf{w} \in \mathcal{W}
\end{array}
$$

- **Linear program formulation without maximum operator:**
  - Introduces auxiliary variable τ to avoid nondifferentiable maximum:

$$
\begin{array}{ll}
\underset{\mathbf{w},\tau}{\text{maximize}} & \mathbf{w}^\mathsf{T}\boldsymbol{\mu} - \lambda\, \tau\\
\text{subject to} & \tau \ge -\mathbf{w}^\mathsf{T}\mathbf{r}_t, \quad t = 1,\dots,T\\
& \mathbf{w} \in \mathcal{W}.
\end{array}
$$

Formulation for the worst-case portfolio

- **Solving the roblem:**
  - The problem is a linear program, assuming $\mathcal{W}$ is defined via linear constraints.
  - Can be efficiently solved using a linear programming solver.

- **Key insights:**
  - The worst-case portfolio formulation provides a direct approach to managing extreme risk by focusing on the most adverse return scenario.
  - This approach is particularly useful for highly risk-averse investors or in markets where extreme losses can have significant impacts.
  - The linear programming formulation ensures that the optimization problem can be solved efficiently, making it practical for real-world portfolio management.

Numerical experiments

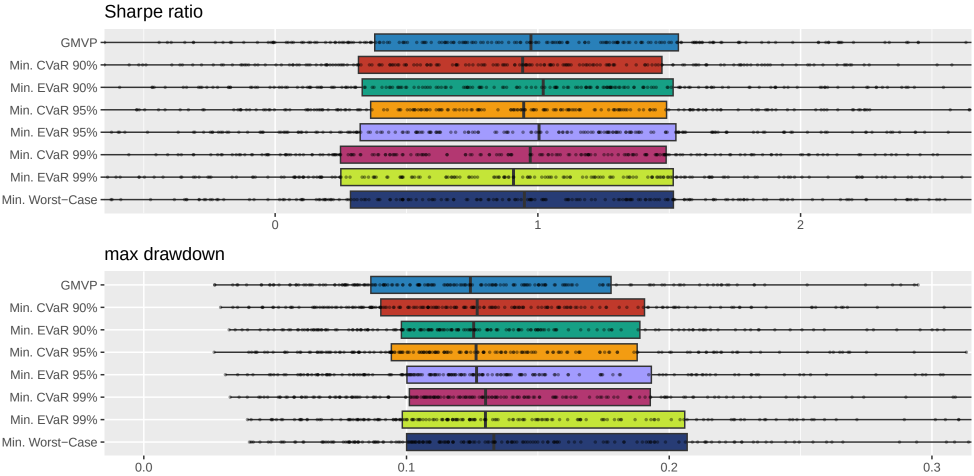
- **Objective:**
  - Compare CVaR, EVaR, and worst-case portfolios based on their respective convex formulations.
  - Exclude expected return from optimization to isolate the impact of risk measures.
  - Use the Global Minimum Variance Portfolio (GMVP) as a benchmark for comparison.
- **Considerations:**
  - Risk measures are non-parametric, relying on observed returns rather than a covariance matrix.
  - Tail events are infrequent, potentially leading to poor characterization of true tail risk.
  - Worst-case portfolio is determined by a single data point, while CVaR and EVaR may suffer from limited tail observations, especially with higher α values.
  - EVaR may offer a more stable characterization by utilizing all observations.
- **Alternative methods:**
  - Parametric models or extreme value theory may provide more stable risk characterizations.

Numerical experiments

- **Setup for comparison:**
  - 200 realizations of 50 randomly selected stocks from the S&P 500 during 2015-2020.
  - Reoptimize portfolios monthly with a one-year lookback period.
- **Metrics for comparison:**
  - Sharpe ratio: Measures risk-adjusted return.
  - Maximum drawdown: Indicates the largest single drop from peak to trough during the observation period.
- **Visualization:**
  - Boxplots in next figure display the distribution of Sharpe ratios and maximum drawdowns for CVaR, EVaR, and worst-case portfolios.
  - This visual comparison helps to understand the trade-offs between risk and return for each risk measure.
- **Preliminary observations:**
  - EVaR portfolios may produce better results than CVaR portfolios, as they consider all observations rather than just the tail.
  - However, drawing definitive conclusions from this numerical experiment is challenging due to the inherent limitations in tail risk characterization.

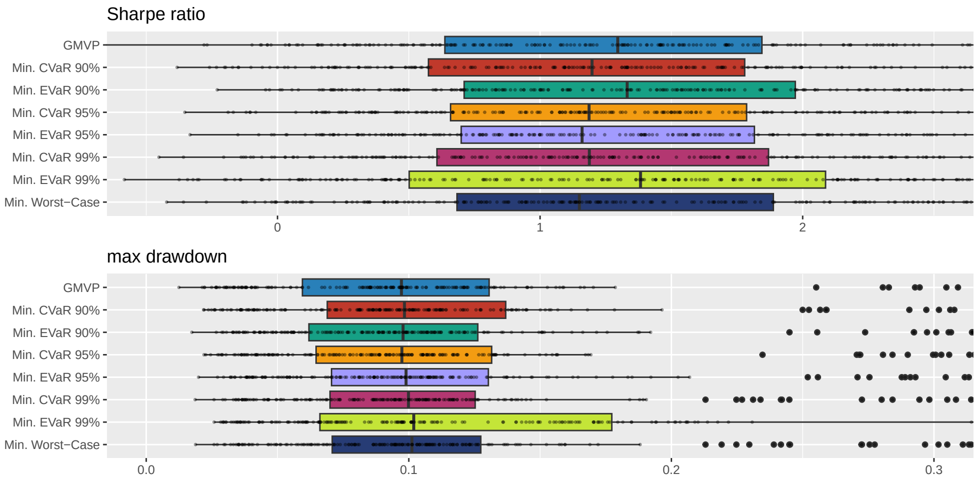
Numerical experiments

Backtest performance of CVaR and EVaR portfolios for dimension $N = 5$:



Numerical experiments

Backtest performance of CVaR and EVaR portfolios for dimension $N = 50$:



Outline

Drawdown portfolios

- **Portfolio return and cumulative return:**
  - Portfolio return at time t: $R_t = \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t$.
  - Cumulative return: $R_t^{\text{cum}} = \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t^{\text{cum}}$.
  - Cumulative returns of assets: $\boldsymbol{r}_t^{\text{cum}} = \sum_{\tau=1}^{t}\boldsymbol{r}_\tau$.
  - Linear or log-returns can be used, corresponding to uncompounded or compounded returns, respectively.

- **Absolute drawdown:**
  - Defined as the difference between the maximum cumulative return up to time t and the cumulative return at time t:

$$D_t(\boldsymbol{w}) = \max_{1\leq\tau\leq t} \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_\tau^{\text{cum}} - \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t^{\text{cum}}.$$

  - Constraint for limiting absolute drawdown to α ($D_t(\boldsymbol{w}) \leq \alpha$):

$$\boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t^{\text{cum}} \geq \max_{1\leq\tau\leq t} \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_\tau^{\text{cum}} - \alpha.$$

Drawdown portfolios

- **Normalized drawdown:**
  - Ratio of the absolute drawdown to the maximum portfolio value up to time t:
$$\bar{D}_t(\boldsymbol{w}) = \frac{\max\limits_{1\leq\tau\leq t} \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_\tau^{\text{cum}} - \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t^{\text{cum}}}{1 + \max\limits_{1\leq\tau\leq t} \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_\tau^{\text{cum}}}.$$
  - Constraint for limiting normalized drawdown to α ($\bar{D}_t(\boldsymbol{w}) \leq \alpha$):
$$\boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t^{\text{cum}} \geq (1-\alpha) \max_{1\leq\tau\leq t} \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_\tau^{\text{cum}} - \alpha.$$
- **Key points:**
  - Drawdown portfolios are developed via statistical models or data-driven methods to minimize drawdowns.
  - Absolute drawdown measures raw loss, while normalized drawdown assesses loss relative to peak portfolio value.
  - Drawdown constraints can be integrated into optimization to manage downside risk.
  - Emphasizing drawdowns in portfolio strategies is crucial for mitigating significant losses and controlling risk.

Formulation for the Max-DD portfolio

- **Mean-Max-DD formulation:**
  - Replaces variance with maximum drawdown (Max-DD) as risk measure:

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \lambda\,\text{Max-DD}(\boldsymbol{w}) \\ \text{subject to} & \boldsymbol{w} \in \mathcal{W} \end{array}$$

- **Substituting Max-DD:**
  - Max-DD is the maximum of drawdowns over time:

$$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \lambda \max\limits_{1\leq t\leq T} \left\{ \max\limits_{1\leq \tau\leq t} \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_\tau^{\text{cum}} - \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t^{\text{cum}} \right\} \\ \text{subject to} & \boldsymbol{w} \in \mathcal{W} \end{array}$$

- **Convexity of the problem:**
  - The problem is convex if $\mathcal{W}$ is convex, as the maximum of convex functions is convex.

Formulation for the Max-DD portfolio

- To get rid of the maximum operators, we first introduce the auxiliary variable s:

$$\begin{array}{ll} \underset{\boldsymbol{w},s}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \lambda\, s \\ \text{subject to} & s \geq \max\limits_{1\leq\tau\leq t} \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_\tau^{\text{cum}} - \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t^{\text{cum}}, \quad t = 1,\dots,T \\ & \boldsymbol{w} \in \mathcal{W} \end{array}$$

- Then, we further introduce the auxiliary variables u_t:

$$\begin{array}{lll} \underset{\boldsymbol{w},\boldsymbol{u},s}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \lambda\, s & \\ \text{subject to} & s \geq u_t - \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t^{\text{cum}}, & t = 1,\dots,T \\ & u_t \geq \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_\tau^{\text{cum}}, & \tau = 1,\dots,t,\ t = 1,\dots,T \\ & \boldsymbol{w} \in \mathcal{W} & \end{array}$$

- Finally, it is possible to simplify the double-index constraints $u_t \geq \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_\tau^{\text{cum}}$ into:

$$\begin{array}{l} u_t \geq \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t^{\text{cum}} \\ u_{t-1} \leq u_t \end{array}, \quad t = 1,\dots,T.$$

Formulation for the Max-DD portfolio

- **Epigraph formulation with auxiliary variables:**
  - Introduces variables s and $\boldsymbol{u}$ to linearize the problem:

$$\begin{array}{ll} \underset{\boldsymbol{w},\boldsymbol{u},s}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \lambda\, s \\ \text{subject to} & \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t^{\text{cum}} \leq u_t \leq s + \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t^{\text{cum}}, \qquad t = 1,\dots,T \\ & u_{t-1} \leq u_t \\ & \boldsymbol{w} \in \mathcal{W} \end{array}$$

- **Sensitivity of Max-DD:**
  - Max-DD is highly sensitive, determined by a single data point.
  - Small changes in portfolio weights or period can lead to different Max-DD values.
  - This sensitivity makes Max-DD a less reliable risk measure.
- **Alternatives to Max-DD:**
  - Average drawdown or conditional drawdown-at-risk can provide more stability.
  - For Gaussian-like distributions, mean-variance framework may suffice.
  - For skewed or heavy-tailed distributions, high-order portfolios are recommended.

Formulation for the Ave-DD portfolio

- **Mean-Ave-DD formulation:**
  - Replaces variance with average drawdown (Ave-DD) as risk measure:

$$
\begin{array}{ll}
\underset{\boldsymbol{w}}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \lambda \frac{1}{T}\sum_{t=1}^{T}\left(\max\limits_{1\leq\tau\leq t} \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_\tau^{\text{cum}} - \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t^{\text{cum}}\right)\\
\text{subject to} & \boldsymbol{w} \in \mathcal{W},
\end{array}
$$

- **Convexity of the problem:**
  - The problem is convex if $\mathcal{W}$ is convex, leveraging the convexity of maximum functions.
- **Epigraph formulation with auxiliary variables:**
  - Introduces variables s and $\boldsymbol{u}$ to linearize the problem:

$$
\begin{array}{ll}
\underset{\boldsymbol{w},\boldsymbol{u},s}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \lambda\, s\\
\text{subject to} & \frac{1}{T}\sum_{t=1}^{T} u_t \leq \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t^{\text{cum}} + s\\
& \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t^{\text{cum}} \leq u_t, \quad t = 1,\dots,T\\
& u_{t-1} \leq u_t\\
& \boldsymbol{w} \in \mathcal{W},
\end{array}
$$

which is a linear program (assuming $\mathcal{W}$ is described by linear constraints).

Formulation for the CVaR-DD portfolio

- **CVaR-DD representation:**
  - CVaR-DD (Conditional Drawdown at Risk) is defined variably as:

$$\text{CVaR-DD}(\boldsymbol{w}) = \inf_{\tau} \left\{ \tau + \frac{1}{1-\alpha} \frac{1}{T} \sum_{t=1}^{T} \left(D_t(\boldsymbol{w}) - \tau\right)^+ \right\}.$$

- **Mean-CVaR-DD formulation:**
  - Incorporates CVaR-DD as a risk measure in portfolio optimization:

$$\begin{array}{ll} \underset{\boldsymbol{w},\tau}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \lambda \left(\tau + \frac{1}{1-\alpha}\frac{1}{T}\sum_{t=1}^{T} \left(\max\limits_{1\leq\tau\leq t} \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_{\tau}^{\text{cum}} - \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_{t}^{\text{cum}} - \tau \right)^+ \right) \\ \text{subject to} & \boldsymbol{w} \in \mathcal{W}, \end{array}$$

Formulation for the CVaR-DD portfolio

- **Convexity and linear program formulation:**
  - The problem is convex if $\mathcal{W}$ is convex.
  - Introduces auxiliary variables s, $\boldsymbol{z}$, and $\boldsymbol{u}$ to linearize the problem:

$$
\begin{array}{ll}
\underset{\boldsymbol{w},\tau,s,\boldsymbol{z},\boldsymbol{u}}{\text{maximize}} & \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \lambda\, s \\
\text{subject to} & s \geq \tau + \frac{1}{1-\alpha}\frac{1}{T}\sum_{t=1}^{T} z_t \\
& 0 \leq z_t \geq u_t - \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t^{\text{cum}} - \tau, \qquad t = 1,\dots,T \\
& \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t^{\text{cum}} \leq u_t \\
& u_{t-1} \leq u_t \\
& \boldsymbol{w} \in \mathcal{W},
\end{array}
$$

- **Drawdown EVaR formulation:**
  - Similar to the EVaR portfolio, a drawdown EVaR can be formulated by replacing loss terms with drawdown measures in the EVaR formulation.

Numerical experiments

- **Portfolio types compared:**
  - Maximum Drawdown (Max-DD) Portfolio
  - Average Drawdown (Ave-DD) Portfolio
  - Drawdown Conditional Value-at-Risk (CVaR-DD) Portfolio
  - Global Minimum Variance Portfolio (GMVP) as a benchmark

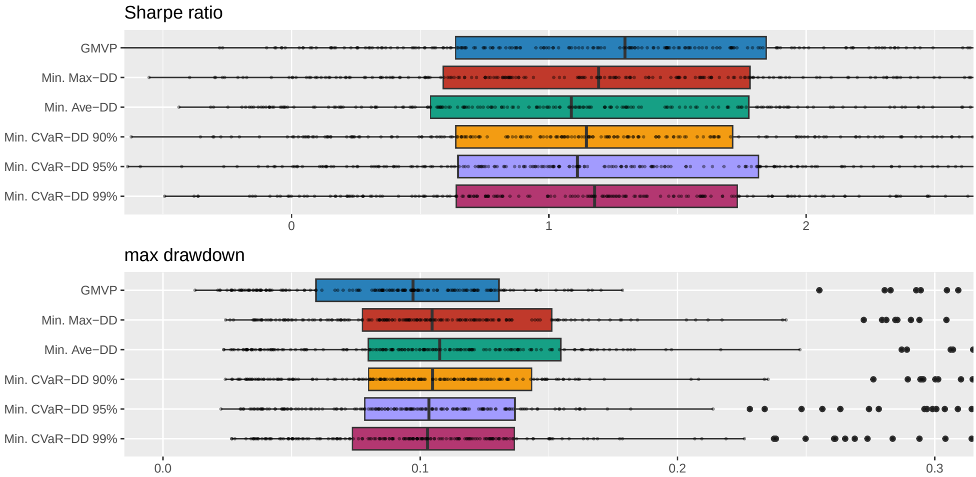
- **Optimization focus:**
  - Risk measures are prioritized by setting $\lambda \to \infty$, excluding expected return from optimization.

- **Caution on drawdown measures:**
  - Drawdown-based measures, especially Max-DD and CVaR-DD, rely on few samples due to the low probability of worst drawdowns.
  - Max-DD is determined by a single sample, while CVaR-DD uses extremely few samples, raising concerns about the reliability of these risk measures.

Numerical experiments

- **Empirical analysis setup:**
  - 200 realizations of portfolios from 50 randomly selected S&P 500 stocks during 2015-2020.
  - Portfolios are reoptimized monthly with a one-year lookback period.
- **Metrics for evaluation:**
  - Sharpe ratio: Assesses risk-adjusted returns.
  - Maximum drawdown: Measures the largest decline from peak to trough.
- **Preliminary observations:**
  - Boxplots in the figure compare the Sharpe ratio and maximum drawdown across portfolio types.
  - Initial results suggest drawdown portfolios do not consistently outperform the GMVP benchmark.
  - Comprehensive empirical testing is needed for definitive conclusions.

Backtest performance of drawdown portfolios:



Numerical experiments: Key insights

- **Drawdown-based portfolio challenges:**
  - The reliance on few extreme samples for drawdown measures introduces significant variability and potential instability in portfolio performance.
- **GMVP as a stable benchmark:**
  - Despite the sophisticated risk management intended by drawdown-based portfolios, the simple GMVP often provides a competitive, if not superior, baseline.
- **Need for further research:**
  - The initial findings highlight the importance of extensive empirical analysis to fully understand the benefits and limitations of drawdown-based portfolio strategies.

Outline

Summary

Variance (or volatility) is a simple risk measure used in Markowitz's 1952 mean-variance modern portfolio theory framework, but since then, more sophisticated measures have been proposed, leading to some notable portfolio formulations:

- **Downside risk portfolios**: The risk focuses on downside losses, formulated in convex form (parameterized by α), with $\alpha = 1$ as a linear program, $\alpha = 2$ as a semivariance portfolio (quadratic program), and $\alpha = 3$ as a more risk-averse convex program.
- **Tail portfolios**: The risk is measured by the tail of the loss distribution, formulated in convex form, including CVaR portfolios (linear program), EVaR portfolios (exponential cone), and worst-case portfolio (linear program).
- **Drawdown portfolios**: The risk is based on drawdown, formulated as linear programs, including maximum drawdown portfolio (single worst drawdown), average drawdown portfolio (average of all drawdowns), and drawdown CVaR portfolio (average of the tail of drawdowns).

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