Portfolio Optimization
Portfolio Basics

Daniel P. Palomar (2024). *Portfolio Optimization: Theory and Application.*
Cambridge University Press.

portfoliooptimizationbook.com

Outline

Abstract

In these slides, we introduce fundamental concepts related to portfolios, including the definition of portfolio weights and the notion of rebalancing. We also discuss common portfolio constraints and performance metrics, as well as a selection of prevalent heuristic and risk-based portfolios employed by practitioners. Examples of these portfolios include the equal-weighted $1/N$ portfolio, the quintile portfolio, and the global minimum variance portfolio (Palomar 2024, chap. 6).

Outline

Data modeling

- **Asset prices and returns:**
  - Denote prices of N assets at time t as $\boldsymbol{p}_t \in \mathbb{R}^N$.
  - Linear returns defined as
    $$\boldsymbol{r}_t^{\text{lin}} = \frac{\boldsymbol{p}_t}{\boldsymbol{p}_{t-1}} - \mathbf{1}.$$
  - Log-returns defined as
    $$\boldsymbol{r}_t^{\text{log}} = \log \boldsymbol{p}_t - \log \boldsymbol{p}_{t-1}.$$

- **Modeling objectives:**
  - Aim to forecast returns at time t using historical data up to $t-1$ ($\mathcal{F}_{t-1}$).
  - Focus on conditional first- and second-order moments (mean vector and covariance matrix).

- **Mean vector and covariance matrix:**
  - Conditional mean vector for log-returns: $\boldsymbol{\mu}_t^{\text{log}} = \mathbb{E}\left[\boldsymbol{r}_t^{\text{log}} \mid \mathcal{F}_{t-1}\right]$.
  - Conditional covariance matrix for log-returns: $\boldsymbol{\Sigma}_t^{\text{log}} = \text{Cov}\left[\boldsymbol{r}_t^{\text{log}} \mid \mathcal{F}_{t-1}\right]$.

Data modeling

- **Preference for log-returns:**
  - Financial models often prefer log-returns for modeling, despite needing linear returns for portfolio performance.
  - Linear and log-returns are approximated as similar for small return values.

- **Approximations between linear and log-returns:**
  - Linear mean and covariance approximated by log counterparts: $\boldsymbol{\mu}_t^{\text{lin}} \approx \boldsymbol{\mu}_t^{\text{log}}$, $\boldsymbol{\Sigma}_t^{\text{lin}} \approx \boldsymbol{\Sigma}_t^{\text{log}}$.
  - Exact mathematical relationships allow deriving linear moments from log moments.

- **Simplification for econometric models:**
  - Time dependency of moments often dropped for simplicity.
  - Linear moments $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ referred to by default, especially under i.i.d. model assumption with no time dependency.

Portfolio

- **Portfolio definition:**
  - Allocation of budget among N risky assets, with uninvested amount as cash.
  - Represented by capital allocation $\boldsymbol{w}_t^{\mathsf{cap}} \in \mathbb{R}^N$, where $w_{i,t}^{\mathsf{cap}}$ is the dollar amount for asset i.
  - Cash represented separately as $c_t^{\mathsf{cap}} \in \mathbb{R}$ or included in $\boldsymbol{w}_t^{\mathsf{cap}}$ as a riskless asset.
  - Portfolio can also be represented by units held $\boldsymbol{w}_t^{\mathsf{units}} \in \mathbb{R}^N$ (e.g., shares or coins).

- **Dollar amount variation:**
  - Keeping unit amount constant, dollar amount changes with asset prices:

  $$\boldsymbol{w}_t^{\mathsf{cap}} = \boldsymbol{w}^{\mathsf{units}} \odot \boldsymbol{p}_t.$$

  - Portfolio time evolution:

  $$\boldsymbol{w}_t^{\mathsf{cap}} = \boldsymbol{w}_{t-1}^{\mathsf{cap}} \odot \left(\mathbf{1} + \boldsymbol{r}_t^{\mathsf{lin}}\right).$$

  - Cash remains constant unless adjusted for contributions or withdrawals.

- **Rebalancing for fixed dollar amount:**
  - To maintain a fixed dollar allocation, portfolios require regular rebalancing.
  - Rebalancing adjusts $\boldsymbol{w}_t^{\mathsf{cap}}$ to original positions, incurring transaction costs.

Evolution of the $1/N$ portfolio over time with effect of rebalancing (vertical lines):

Weight allocation over time for portfolio 1/N

NAV and return
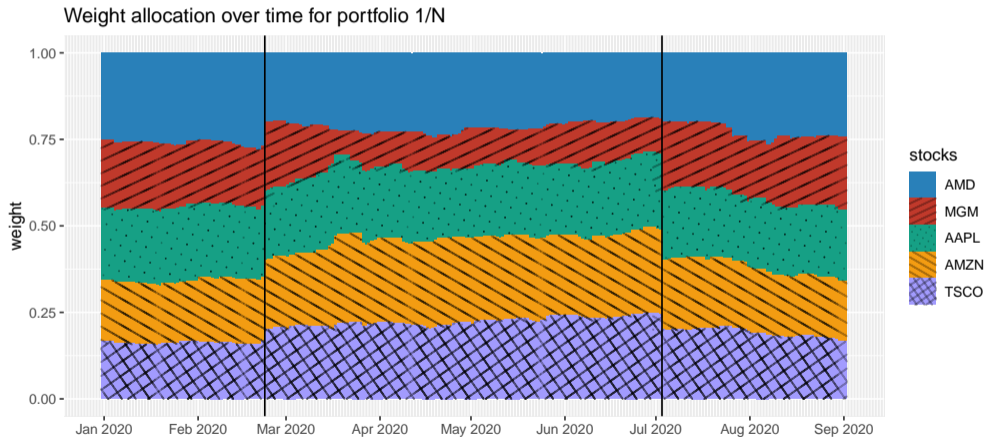
- **Portfolio NAV definition:**
  - The portfolio *net asset value* (NAV) at time t includes current market valuation and cash:

$$\text{NAV}_t \triangleq \mathbf{1}^\mathsf{T} \boldsymbol{w}_t^{\text{cap}} + c_t^{\text{cap}}.$$

- **NAV evolution:**
  - Evolution derived from portfolio time evolution:

$$\text{NAV}_t = \text{NAV}_{t-1} + \left(\boldsymbol{w}_{t-1}^{\text{cap}}\right)^\mathsf{T} \boldsymbol{r}_t^{\text{lin}}.$$

  - Indicates NAV change depends on assets' returns.
- **Portfolio return:**
  - Defined as the change in NAV relative to previous NAV:

$$R_t^{\text{portf}} \triangleq \frac{\text{NAV}_t - \text{NAV}_{t-1}}{\text{NAV}_{t-1}} = \boldsymbol{w}_{t-1}^\mathsf{T} \boldsymbol{r}_t^{\text{lin}}.$$

NAV and return

- **Normalized portfolio:**
  - Portfolio normalized with respect to current NAV:

  $$\boldsymbol{w}_t = \boldsymbol{w}_t^{\mathsf{cap}}/\mathsf{NAV}_t$$

  - Used in portfolio design and optimization.

- **Asset additivity property:**
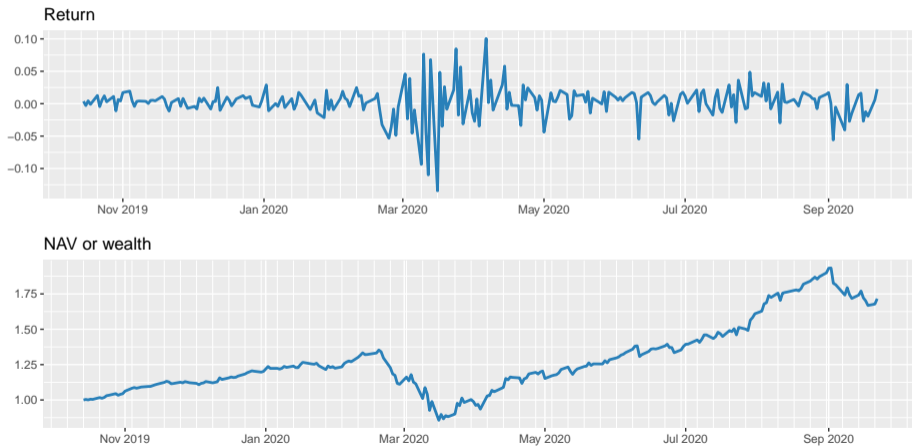 - Linear returns exhibit asset additivity, crucial for portfolio return calculation.

- **Adjustment for cash contributions:**
  - If cash contributions occur, adjust portfolio return calculation:

  $$R_t^{\mathsf{portf}} \triangleq \frac{\mathsf{NAV}_t - \mathsf{NAV}_{t-1} - \mathsf{contribution}_{t-1}}{\mathsf{NAV}_{t-1}} = \boldsymbol{w}_{t-1}^{\mathsf{T}} \boldsymbol{r}_t^{\mathsf{lin}}.$$

NAV and return

Return and NAV of the $1/N$ portfolio over time:

Cumulative return and NAV

- **Cumulative return and NAV:**
  - Cumulative return: $\text{NAV}_t/\text{NAV}_0 - 1$.
  - Does not include contributions or redemptions.
  - Cumulative return plot starts at zero; NAV plot starts at 1 (if normalized).

- **Importance of position size:**
  - Cumulative return and NAV depend on portfolio design and invested budget.
  - Budget may include cash reserves.
  - Capital invested is referred to as *position size*.

Cumulative return and sizing

- **Full reinvesting:**
  - Normalized portfolio $\boldsymbol{w}_t$ with full reinvestment of current NAV: $\boldsymbol{w}_t^{\mathsf{cap}} = \mathsf{NAV}_t \times \boldsymbol{w}_t$.
  - NAV evolution:
  $$\mathsf{NAV}_t = \mathsf{NAV}_{t-1} \times \left(1 + R_t^{\mathsf{portf}}\right)$$

  - Geometric growth:
  $$\mathsf{NAV}_t = \mathsf{NAV}_0 \times \left(1 + R_1^{\mathsf{portf}}\right) \times \cdots \times \left(1 + R_t^{\mathsf{portf}}\right)$$

- **Constant reinvesting:**
  - Normalized portfolio $\boldsymbol{w}_t$ with constant reinvestment of initial NAV: $\boldsymbol{w}_t^{\mathsf{cap}} = \mathsf{NAV}_0 \times \boldsymbol{w}_t$.
  - NAV evolution:
  $$\mathsf{NAV}_t = \mathsf{NAV}_{t-1} + \mathsf{NAV}_0 \times R_t^{\mathsf{portf}}$$

  - Arithmetic growth:
  $$\mathsf{NAV}_t = \mathsf{NAV}_0 \times \left(1 + R_1^{\mathsf{portf}} + R_2^{\mathsf{portf}} + \cdots + R_t^{\mathsf{portf}}\right)$$

Cumulative return

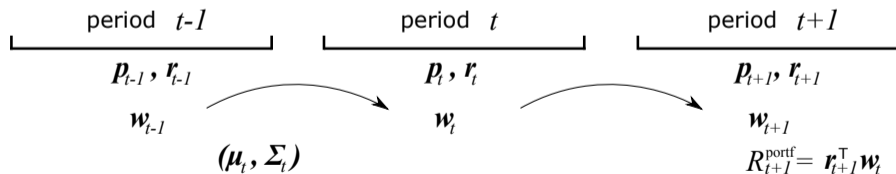Comparison of arithmetic and geometric cumulative returns:

Cumulative return or P&L

arithmetic
geometric
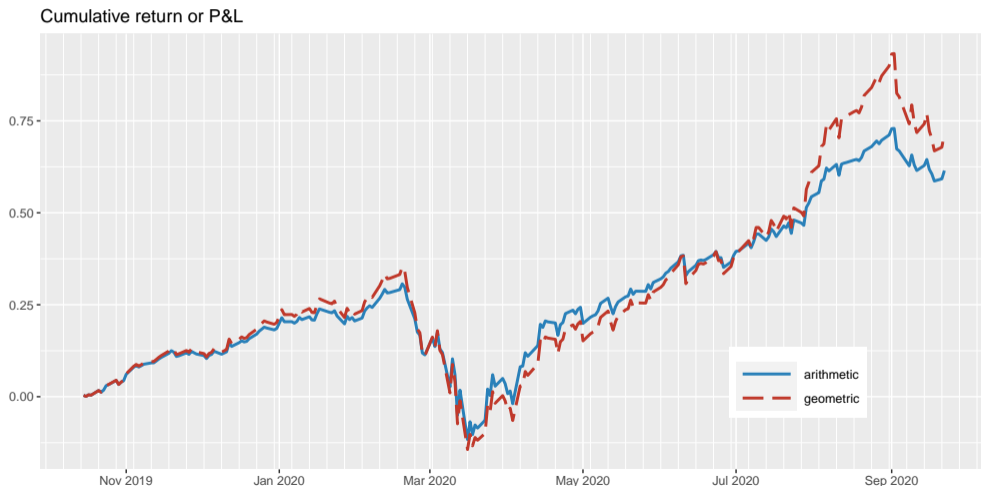
- **Portfolio time notation:**
  - Portfolio at time t is denoted as $\boldsymbol{w}_t$.
  - $\boldsymbol{w}_t$ uses information up to time $t-1$ (moments $\boldsymbol{\mu}_t$ and $\boldsymbol{\Sigma}_t$).
  - Executed at time t, assuming instantaneous execution.

- Portfolio notation over time periods:

- **Incorrect return calculation:**
  - Multiplying $\boldsymbol{w}_t$ with return at the same period $\boldsymbol{r}_t^{\text{lin}}$ is incorrect.
  - This would incur look-ahead bias (refer to the topic of backtesting and its dangers).

- **Correct return calculation:**
  - $\boldsymbol{w}_t$ is the portfolio executed and held at time t, with prices $\boldsymbol{p}_t$.
  - Its return can be computed upon observing prices $\boldsymbol{p}_{t+1}$.
  - Correct return calculation: $\boldsymbol{w}_t^{\mathsf{T}} \boldsymbol{r}_{t+1}^{\text{lin}}$.

- **Alternative time index definition:**
  - Some authors define returns with a shift in the time index.
  - This allows $\boldsymbol{p}_t$ to be multiplied with $\boldsymbol{r}_t^{\text{lin}}$.
  - Care must be taken to maintain consistency in notation and avoid look-ahead bias.

Portfolio notation over time[*]

- **First- and second-order moments in portfolio optimization:**
  - Expected value and variance are crucial for making informed investment decisions.
  - Optimization techniques aim to find portfolios with desirable expected returns and risk levels.

- **Expected value of portfolio return:**
  - Given by:
  $$\mathbb{E}\left[R_t^{\text{portf}}\right] = \boldsymbol{w}_{t-1}^{\mathsf{T}}\boldsymbol{\mu}$$
  - Represents the mean return of the portfolio.

- **Variance of portfolio return:**
  - Given by:
  $$\text{Var}\left[R_t^{\text{portf}}\right] = \boldsymbol{w}_{t-1}^{\mathsf{T}}\boldsymbol{\Sigma}\boldsymbol{w}_{t-1}$$
  - Represents the risk or volatility of the portfolio return.

Transaction costs

- **Transaction costs overview:**
  - Consist of commission fee and slippage.
  - Commission fee: Payment to broker, varies by country and broker.
  - Slippage: Difference between expected and actual execution price, affects liquid and illiquid assets differently.

- **Impact on portfolio NAV:**
  - Rebalancing to $\boldsymbol{w}_t^{\text{cap,reb}}$ decreases NAV by transaction cost $\phi(\boldsymbol{w}_t^{\text{cap}} \to \boldsymbol{w}_t^{\text{cap,reb}})$.
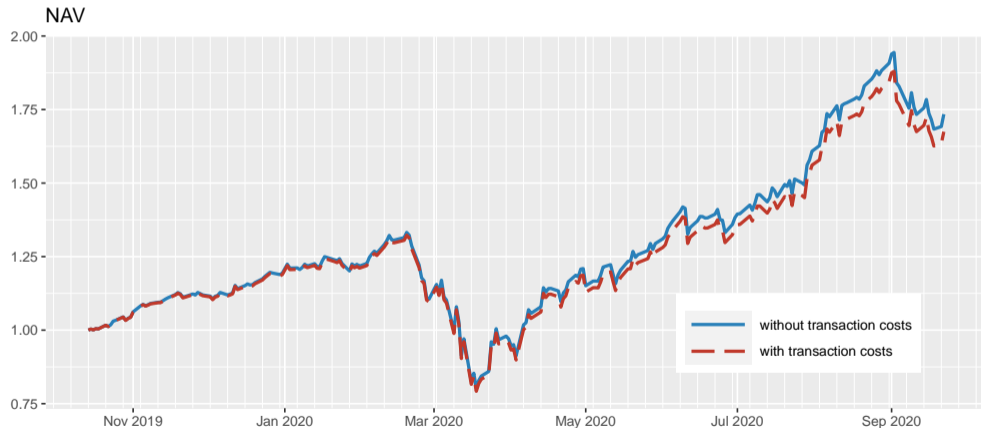 - NAV after rebalancing and price change:

  $$\text{NAV}_t = \text{NAV}_{t-1} + \left(\boldsymbol{w}_{t-1}^{\text{cap,reb}}\right)^{\mathsf{T}} \boldsymbol{r}_t^{\text{lin}} - \phi\left(\boldsymbol{w}_{t-1}^{\text{cap}} \to \boldsymbol{w}_{t-1}^{\text{cap,reb}}\right).$$

  - Portfolio return with transaction costs:

  $$R_t^{\text{portf}} = \left(\boldsymbol{w}_{t-1}^{\text{reb}}\right)^{\mathsf{T}} \boldsymbol{r}_t^{\text{lin}} - \phi\left(\boldsymbol{w}_{t-1} \to \boldsymbol{w}_{t-1}^{\text{reb}}\right).$$

Transaction costs

Effect of transaction costs on the NAV (with daily rebalancing and 90 bps of fees):

- **Expected value and variance with transaction costs:**
  - Expected value:
  $$\mathbb{E}\left[R_t^{\text{portf}}\right] = \left(\boldsymbol{w}_{t-1}^{\text{reb}}\right)^{\mathsf{T}}\boldsymbol{\mu} - \phi\left(\boldsymbol{w}_{t-1} \rightarrow \boldsymbol{w}_{t-1}^{\text{reb}}\right).$$

  - Variance:
  $$\text{Var}\left[R_t^{\text{portf}}\right] = \left(\boldsymbol{w}_{t-1}^{\text{reb}}\right)^{\mathsf{T}}\boldsymbol{\Sigma}\boldsymbol{w}_{t-1}^{\text{reb}}.$$

Transaction costs[*]

- **Transaction cost function:**
  - Decomposed into fees and slippage:

  $$\phi\left(\boldsymbol{w}_t \to \boldsymbol{w}_t^{\text{reb}}\right) = \phi^{\text{fees}}\left(\boldsymbol{w}_t \to \boldsymbol{w}_t^{\text{reb}}\right) + \phi^{\text{slippage}}\left(\boldsymbol{w}_t \to \boldsymbol{w}_t^{\text{reb}}\right).$$

  - Fees proportional to turnover:

  $$\phi^{\text{fees}}\left(\boldsymbol{w}_t \to \boldsymbol{w}_t^{\text{reb}}\right) \approx \tau^{\text{fee}}\|\boldsymbol{w}_t^{\text{reb}} - \boldsymbol{w}_t\|_1.$$

  - Slippage varies by asset:

  $$\phi^{\text{slippage}}\left(\boldsymbol{w}_t \to \boldsymbol{w}_t^{\text{reb}}\right) \approx \sum_{i=1}^{N} \tau_i^{\text{slippage}}|w_{it}^{\text{reb}} - w_{it}|.$$

- **Practical considerations:**
  - Account for transaction costs in backtests for realistic performance assessment.
  - Monitor strategy turnover and return on turnover due to transaction cost implications.

Portfolio rebalancing

- **Portfolio rebalancing trade-off:**
  - Frequent rebalancing aligns the market portfolio with the desired one.
  - Less frequent rebalancing reduces transaction costs.

- **Rebalancing schemes:**
  - Calendar-based: Daily, weekly, or monthly rebalancing.
  - Adaptive: Rebalance only if portfolio distance exceeds a threshold.
  - Measures of difference: Turnover, Mahalanobis distance, tracking error squared distance.

- **Turnover considerations:**
  - Turnover is often upper-bounded or penalized to avoid excessive rebalancing.

Portfolio rebalancing[*]

Portfolio notation over time periods with rebalancing:
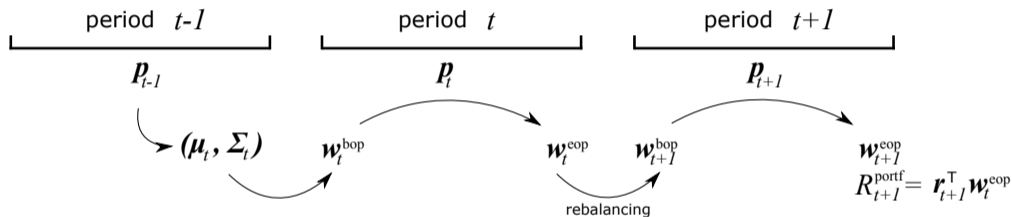
- **Time-index notation for rebalancing:**
  - Period t divided into beginning of period (bop) and end of period (eop).
  - Portfolio w_t separated into w_t^{bop} and w_t^{eop}.

Portfolio rebalancing*

- **Price change and rebalancing:**
  - Price change:
  $$\boldsymbol{w}_t^{\text{eop}} = \boldsymbol{w}_t^{\text{bop}} \odot \left(\boldsymbol{1} + \boldsymbol{r}_t^{\text{lin}}\right).$$
  - Rebalancing incurs transaction costs; if none, then $\boldsymbol{w}_{t+1}^{\text{bop}} = \boldsymbol{w}_t^{\text{eop}}$.

- **Fixed portfolio rebalancing:**
  - After rebalancing (without transaction costs):
  $$\boldsymbol{w}_t^{\text{bop}} = \boldsymbol{w}$$
  - After price change:
  $$\boldsymbol{w}_t^{\text{eop}} = \boldsymbol{w} \odot \left(\boldsymbol{1} + \boldsymbol{r}_t^{\text{lin}}\right).$$
  - Portfolio return:
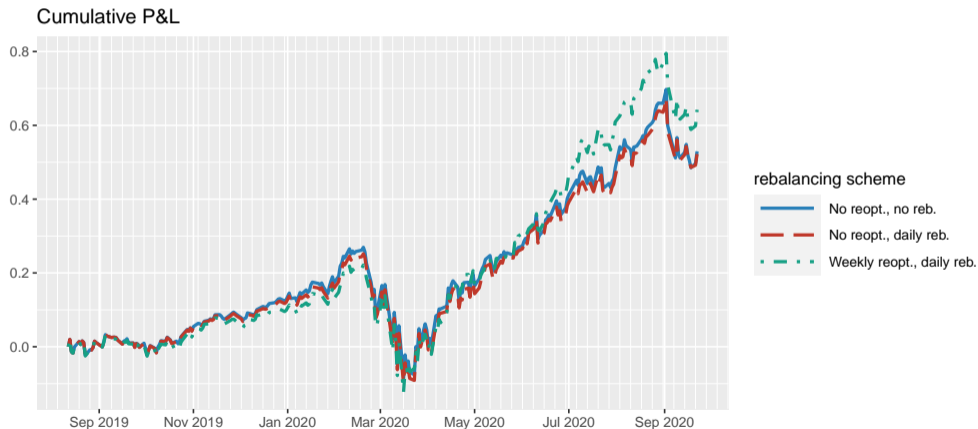 $$R_t^{\text{portf}} = \boldsymbol{w}^{\mathsf{T}} \boldsymbol{r}_t^{\text{lin}}.$$

- **Effect of reoptimization and rebalancing:**
  - Different schemes affect performance, especially with transaction costs.
  - Rebalancing vs. Reoptimization: Execution in the market vs. updating portfolio design based on new data.

Portfolio rebalancing*

Backtest cumulative P&L of a portfolio with different reoptimization and rebalancing schemes:

Cumulative P&L

rebalancing scheme
— No reopt., no reb.
— — No reopt., daily reb.
-·- Weekly reopt., daily reb.

Outline

Portfolio constraints

- **Common portfolio constraints:**
  - Imposed by regulators or brokers: Shorting constraints, leverage constraints, margin requirements.
  - Optional (investor's views): Market neutrality, portfolio sparsity, diversity enforcement.
  - Basic reference: (Palomar 2024, chap. 6).

- **Examples:**
  - long-only or no-shorting constraint
  - capital budget constraint
  - holding constraints
  - cardinality constraint
  - turnover constraint
  - market-neutral constraint
  - dollar-neutral constraint
  - diversification constraint
  - leverage constraint
  - margin requirements

Portfolio constraints

- **Long-only or no-shorting constraint:**
  - Investors typically take long positions.
  - Short selling allowed in some markets, implying negative elements in **w**.
  - Constraint for no shorting:

$$\boldsymbol{w} \geq \boldsymbol{0},$$

  which is linear and convex.

- **Capital budget constraint:**
  - Without shorting or leverage:

$$\boldsymbol{1}^\mathsf{T}\boldsymbol{w} + c = 1,$$

  linear and convex.
  - Without cash variable:

$$\boldsymbol{1}^\mathsf{T}\boldsymbol{w} \leq 1.$$

- If equality is imposed, $\boldsymbol{1}^\mathsf{T}\boldsymbol{w} = 1$, the implication is that the cash is zero and the portfolio is fully invested in the risky assets.
- Implies full investment in risky assets if equality is imposed (cash is zero).

Portfolio constraints

- **Holding constraints:**
    - Limits on maximum positions for diversification
    - May include minimum positions for desired asset holdings.
    - Holding constraints are imposed via lower and upper bounds:

    $$\boldsymbol{l} \leq \boldsymbol{w} \leq \boldsymbol{u},$$

    which is linear and convex.

- **Cardinality constraint:**
    - Limits the number of active positions to simplify operations:

    $$\|\boldsymbol{w}\|_0 \leq K,$$

    nonconvex and challenging to handle.

- **Turnover constraint:**
    - Controls portfolio turnover to manage transaction costs:

    $$\|\boldsymbol{w} - \boldsymbol{w}_0\|_1 \leq u,$$

    convex as norms are convex functions.

Portfolio constraints

- **Market-neutral constraint:**
  - Avoids market exposure, ensuring portfolio is orthogonal to market "beta":

$$\beta^{\mathsf{T}} \boldsymbol{w} = \boldsymbol{0},$$

  linear and convex.

- **Dollar-neutral constraint:**
  - Balances long and short positions:

$$\boldsymbol{1}^{\mathsf{T}} \boldsymbol{w} = \boldsymbol{0}.$$

  - Should be used with a leverage constraint due to practical limitations on short selling.

- **Diversification constraint:**
  - Promotes allocation across multiple assets:

$$\|\boldsymbol{w}\|_2^2 \leq D,$$

  where $1/N \leq D < 1$.
  - Uses the Herfindahl index as a diversification measure, convex constraint.

Portfolio constraints

- **Leverage constraint:**
  - Limits borrowing for long and short positions
  - The ℓ_1-norm measures the total amount of long and short positions:

$$\|\boldsymbol{w}\|_1 \leq L,$$

  where L indicates the leverage.

- **Margin requirements:**
  - Brokers impose margin requirements for borrowing:

$$m^{\text{long}} \times \mathbf{1}^{\mathsf{T}} \boldsymbol{w}^+ + m^{\text{short}} \times \mathbf{1}^{\mathsf{T}} \boldsymbol{w}^- \leq 1,$$

  - Separates portfolio into longs ($\boldsymbol{w}^+$) and shorts ($\boldsymbol{w}^-$) with specific margin requirements for each.

Outline

Performance measures

- **Portfolio performance assessment:**
  - Recall the expression of the portfolio return:

  $$R_t^{\text{portf}} = \boldsymbol{w}^{\mathsf{T}} \boldsymbol{r}_t$$

  - All the performance measures are derived from the portfolio return.
  - References: (Bacon 2008), (Palomar 2024, chap. 6).

- **Examples:**
  - expected return
  - volatility
  - Sharpe ratio and information ratio
  - downside risk and semivariance
  - gain-loss ratio
  - Sortino ratio
  - Value-at-risk (VaR) and Conditional value-at-risk (CVaR)
  - drawdown
  - Calmar ratio and Sterling ratio

- **Expected return:**
  - One-period expected return:

  $$\mathbb{E}\left[R_t^{\text{portf}}\right] = \boldsymbol{w}^{\mathsf{T}}\boldsymbol{\mu}.$$

  - Reflects average investment benefit, excluding risk.

- **Annualized return:**
  - Scaled by number of periods per year, γ:

  $$\text{Annualized Return} = \gamma \times \boldsymbol{w}^{\mathsf{T}}\boldsymbol{\mu}.$$

  - For daily prices: $\gamma = 252$ (stock market) or $\gamma = 365$ (cryptocurrency market).
  - For hourly prices: $\gamma = 8,760$ (cryptocurrency market).

Performance measures: Expected return

- **Sample mean annualized return:**
  - Arithmetic chaining (assuming T returns):

$$\frac{\gamma}{T} \sum_{t=1}^{T} R_t^{\text{portf}}.$$

- **Compound annual growth rate (CAGR):**
  - Geometric chaining for compounding:

$$\left(\prod_{t=1}^{T} \left(1 + R_t^{\text{portf}} \right) \right)^{\gamma/T} - 1.$$

  - Geometric mean is lower than arithmetic mean before annualization.

- **Volatility:**
  - Standard deviation of portfolio return:

  $$\text{Std}\left[R_t^{\text{portf}}\right] = \sqrt{\boldsymbol{w}^{\mathsf{T}}\boldsymbol{\Sigma}\boldsymbol{w}}.$$

  - Simplest risk measure, variance used by Markowitz.

- **Annualized volatility:**
  - Scaled by $\sqrt{\gamma}$:

  $$\text{Annualized Volatility} = \sqrt{\gamma} \times \sqrt{\boldsymbol{w}^{\mathsf{T}}\boldsymbol{\Sigma}\boldsymbol{w}}.$$

- **Volatility-adjusted returns:**
  - Adjusts returns for volatility comparison:

$$\bar{R}_t^{\text{portf}} = \boldsymbol{w}^{\mathsf{T}} \boldsymbol{r}_t \times \frac{\sigma^{\text{target}}}{\sigma_t},$$

  - σ_t: Volatility of $\boldsymbol{w}^{\mathsf{T}} \boldsymbol{r}_t$.
  - σ^{target}: Desired target volatility.

- **Cumulative returns comparison:**
  - Non-adjusted may mislead due to different volatilities.
  - Volatility-adjusted provides a better return-volatility trade-off.
  - Sharpe ratio captures this trade-off.

Cumulative returns and volatility-adjusted cumulative returns:
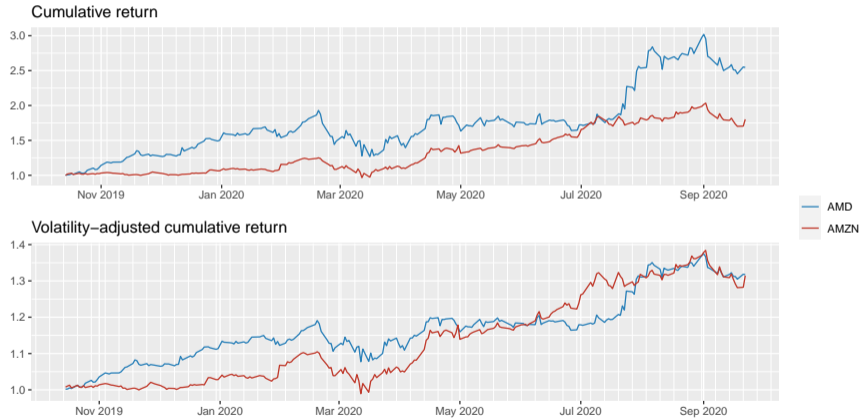
Performance measures: Sharpe ratio

- **Sharpe ratio (SR):**
  - Risk-adjusted expected excess return:

  $$SR = \frac{\mathbf{w}^{\mathsf{T}}\boldsymbol{\mu} - r_{\mathsf{f}}}{\sqrt{\mathbf{w}^{\mathsf{T}}\boldsymbol{\Sigma}\mathbf{w}}},$$

  - r_{f}: risk-free rate.
  - Measures return per unit of risk.

- **Annualized Sharpe ratio:**
  - Adjusted for number of periods per year:

  $$\text{Annualized SR} = \sqrt{\gamma} \times \frac{\mathbf{w}^{\mathsf{T}}\boldsymbol{\mu} - r_{\mathsf{f}}}{\sqrt{\mathbf{w}^{\mathsf{T}}\boldsymbol{\Sigma}\mathbf{w}}}.$$

Performance measures: Sharpe ratio

- **Geometric Sharpe ratio:**
  - Uses compounded annualized return in the numerator.
  - Based on geometric mean of returns.

- **Rolling Sharpe ratio:**
  - SR computed on a rolling-window basis.
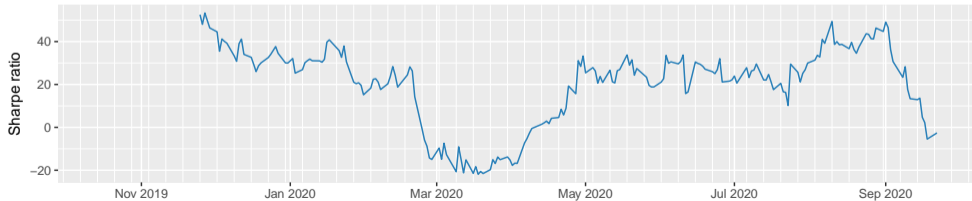 - Useful for assessing performance over time.

Cumulative returns and rolling annualized Sharpe ratio (overall SR of 1.65):

Performance measures: Information ratio

- **Information ratio (IR):**
  - Compares portfolio return to a benchmark:

$$\mathsf{IR} = \frac{\mathbb{E}\left[\boldsymbol{w}^{\mathsf{T}}\boldsymbol{r}_t - r_t^{\mathsf{b}}\right]}{\mathsf{Std}\left[\boldsymbol{w}^{\mathsf{T}}\boldsymbol{r}_t - r_t^{\mathsf{b}}\right]},$$

  - r_t^{b}: return of the benchmark.
  - Measures excess return per unit of risk relative to the benchmark.

- **Downside risk and semivariance:**
  - Variance measures deviation from mean return:

  $$\text{Var}\left[\boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t\right] = \mathbb{E}\left[\left(\boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t - \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu}\right)^2\right].$$

  - Downside risk focuses on underperformance relative to a reference level.

- **Semivariance:**
  - Measures risk of return falling below the mean:

  $$\text{SemiVar}\left[\boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t\right] = \mathbb{E}\left[\left(\left(\boldsymbol{w}^\mathsf{T}\boldsymbol{\mu} - \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t\right)^+\right)^2\right],$$

  - $(\cdot)^+$ avoids penalizing above-mean performance.

Performance measures: Downside risk and semivariance[*]

- **Semi-deviation/downside deviation:**
  - Square root of semivariance, analogous to volatility.

- **Lower partial moments (LPM):**
  - General downside risk measure:

$$\mathsf{LPM}_\alpha(\tau) = \mathbb{E}\left[\left(\left(\tau - \boldsymbol{w}^\mathsf{T}\boldsymbol{r}_t\right)^+\right)^\alpha\right],$$

  - τ: disaster level.
  - α: reflects investor's risk attitude (risk-averse, neutral, risk-seeking).
  - Can form various downside measures by choosing α and τ (e.g., semivariance with $\alpha = 2$ and $\tau = \boldsymbol{w}^\mathsf{T}\boldsymbol{\mu}$).

- **Gain-loss ratio (GLR):**
  - Downside risk measure comparing positive and negative returns.
  - Formula:
    $$\mathsf{GLR} = \frac{\mathbb{E}\left[R_t^{\mathrm{portf}} \mid R_t^{\mathrm{portf}} > 0\right]}{-\mathbb{E}\left[R_t^{\mathrm{portf}} \mid R_t^{\mathrm{portf}} < 0\right]}.$$
  - Reflects the average gain to average loss ratio.
  - Note: The denominator is made positive for a meaningful ratio by taking its absolute value.

- **Sortino Ratio:**
  - Adjusts for downside risk instead of total volatility.
  - Formula:
    $$\text{SoR} = \frac{\boldsymbol{w}^{\mathsf{T}}\boldsymbol{\mu} - r_{\mathsf{f}}}{\sqrt{\text{SemiVar}\left[\boldsymbol{w}^{\mathsf{T}}\boldsymbol{r}_t\right]}},$$
  - Focuses on undesirable volatility (downside risk).
  - More appropriate for investors concerned primarily with downside risk.

Performance measures: Value-at-risk (VaR)

- **Value-at-Risk (VaR):**
  - VaR focuses on the tail of the distribution, indicating potential large losses.
  - Measures maximum loss at a specified confidence level (e.g., 95%).
  - Defined as the quantile of the loss distribution:

  $$\text{VaR}_\alpha = \inf \{\xi_0 \mid \Pr(\xi \leq \xi_0) \geq \alpha\},$$

  where $\xi_t = -\boldsymbol{w}^\mathsf{T} \boldsymbol{r}_t$ represents the loss.
  - α is the confidence level (e.g., $\alpha = 0.95$ for 95% confidence).

- **Drawbacks of VaR:**
  - Does not quantify the shape of the tail beyond the VaR threshold.
  - Cannot indicate the magnitude of losses exceeding the VaR.
  - Not subadditive, challenging the principle of diversification (merging two portfolios can increase perceived risk).

Illustration of distribution of returns and measures of risk:
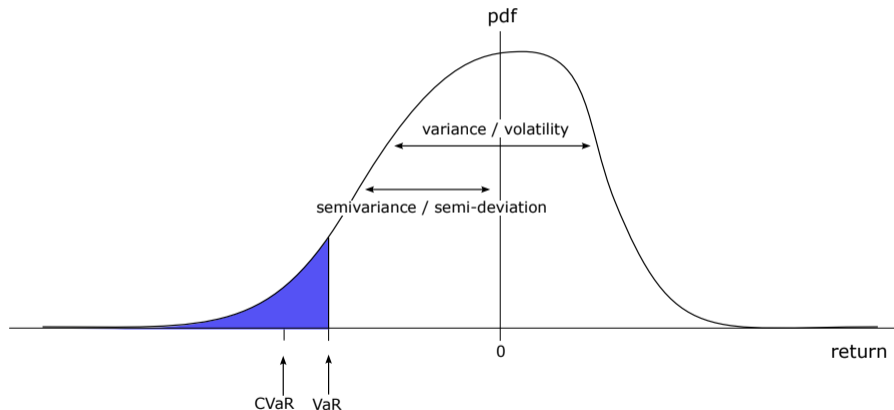
- **Conditional Value-at-Risk (CVaR):**
  - Expected value of the loss tail:

  $$\mathsf{CVaR}_\alpha = \mathbb{E}\left[\xi \mid \xi \geq \mathsf{VaR}_\alpha\right].$$

  - Also known as Expected Shortfall (ES).
  - Takes into account the average value of losses exceeding the VaR.

- **Advantages of CVaR:**
  - Provides insight into the magnitude of losses beyond the VaR threshold.
  - Subadditive, aligning with the principle of diversification.

Performance measures: Drawdown

- **Drawdown (DD):**
  - Measures decline from a historical peak in cumulative profit or NAV:

$$D_t = \mathrm{HWM}_t - \mathrm{NAV}_t,$$

  - High Water Mark (HWM):

$$\mathrm{HWM}_t = \max_{\tau \in [0,t]} \mathrm{NAV}_\tau.$$

- **Normalized drawdown:**
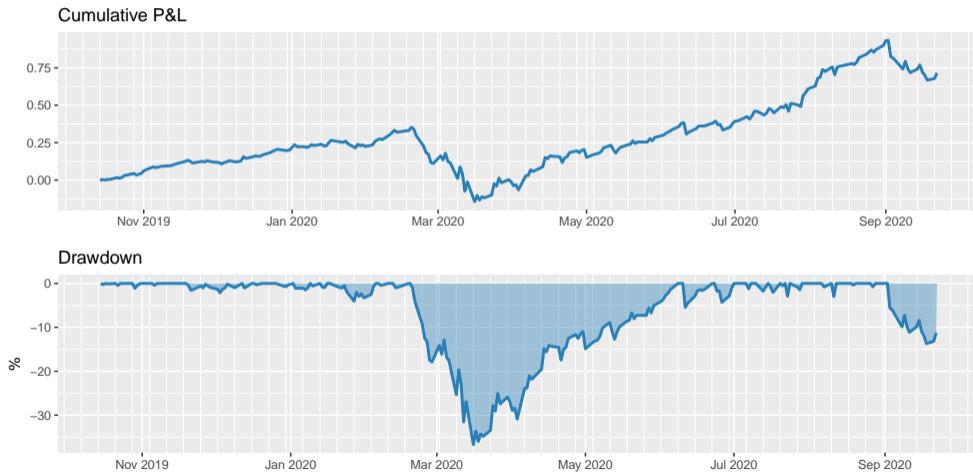 - Relative decline from the peak:

$$\bar{D}_t = \frac{\mathrm{HWM}_t - \mathrm{NAV}_t}{\mathrm{HWM}_t}.$$

- **Drawdown as a path-dependent measure:**
  - Depends on the sequence of returns, unlike other performance measures.

Performance measures: Drawdown

Cumulative P&L and corresponding drawdown of a portfolio:

Performance measures: Drawdown[*]

- **Maximum drawdown (Max-DD):**
  - Largest observed drawdown:
  $$\text{Max-DD} = \max_{1 \leq t \leq T} \bar{D}_t;$$

- **Average drawdown (Ave-DD):**
  - Mean of drawdowns over time:
  $$\text{Ave-DD} = \frac{1}{T} \sum_{1 \leq t \leq T} \bar{D}_t;$$

- **Conditional drawdown-at-risk (CDaR):**
  - Expected drawdown in the worst cases:
  $$\text{CDaR}_\alpha = \mathbb{E}\left[\bar{D}_t \mid \bar{D}_t \geq \text{VaR}_\alpha\right],$$

  - VaR_α: Value at risk of drawdown at confidence level α.

Performance measures: Calmar ratio and Sterling ratio[*]

- **Calmar Ratio:**
  - Adjusts Sharpe ratio by using maximum drawdown:

  $$\text{Calmar ratio} = \frac{\mathbf{w}^{\mathsf{T}}\boldsymbol{\mu} - r_{\mathsf{f}}}{\text{Max-DD}}.$$

  - Focuses on return per unit of maximum drawdown risk.

- **Sterling Ratio:**
  - Similar to Calmar ratio, with an added risk measure:

  $$\text{Sterling ratio} = \frac{\mathbf{w}^{\mathsf{T}}\boldsymbol{\mu} - r_{\mathsf{f}}}{\text{Max-DD} + 0.1}.$$

  - Incorporates an excess risk measure to account for additional risk considerations.

Outline

Heuristic portfolios

- **Heuristic portfolios overview:**
  - Not mathematically derived but widely used in practice.
  - Serve as benchmarks for portfolio performance.
  - Include simple, rule-based strategies for asset allocation.

- **Examples:**
  - buy&hold (B&H)
  - global maximum return portfolio (GMRP)
  - $1/N$ portfolio
  - quintile portfolios

- **Basic reference:** (Palomar 2024, chap. 6).

- **Buy&Hold (B&H) Portfolio:**
  - Simple investment strategy focusing on a single asset or a set of assets.
  - Allocation:

  $$\boldsymbol{w} = \boldsymbol{e}_i,$$

  where $\boldsymbol{e}_i$ is an all-zero vector except for a one at the ith element.
  - Lacks diversification in its single-asset form.
  - Aimed at long-term investment without adjustments over short horizons.
  - Based on the belief that the chosen asset(s) will appreciate over time.
  - Asset selection can be guided by fundamental or technical analysis.

- **Global maximum return portfolio (GMRP):**
  - Formulation:

    $$\begin{array}{ll} \underset{\boldsymbol{w}}{\text{maximize}} & \boldsymbol{w}^{\mathsf{T}}\boldsymbol{\mu} \\ \text{subject to} & \mathbf{1}^{\mathsf{T}}\boldsymbol{w} = 1, \quad \boldsymbol{w} \geq \mathbf{0}, \end{array}$$

  - Problem can be solved with an LP solver.
  - However, in this case the solution is trivial: It allocates all the budget to the asset with the highest forecast return.

- **Cautions with GMRP:**
  - Lacks diversification, potentially leading to poor performance.
  - Past performance is not indicative of future results.
  - Estimation of $\boldsymbol{\mu}$ is often noisy and unreliable.

- **Equally weighted portfolio (EWP) or $1/N$ portfolio:**
  - Allocates capital equally across all N assets:

  $$\boldsymbol{w} = \frac{1}{N} \times \boldsymbol{1},$$

  where $\boldsymbol{1} \in \mathbb{R}^N$ is the all-one vector.
  - Emphasizes diversification: "do not put all your eggs in the same basket."

- **Historical context and performance:**
  - Also known as "Talmudic rule" based on ancient recommendation.
  - Has shown superior historical performance and inspired equally weighted ETFs.
  - S&P 500 equally weighted indices developed by Standard & Poor.
  - Claims of outperforming Markowitz's mean-variance portfolio, though debated.

- **Quintile portfolio:**
  - Allocates equally to top 20% of assets based on performance ranking.
  - Can be adjusted for different fractions (e.g., quartile, decile).
  - Long-short version involves short-selling bottom-ranked assets.

- **Portfolio composition:**
  - Assets ranked from top to bottom performers.
  - Allocation:

$$\boldsymbol{w} = \frac{1}{N/5} \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} 20\% \atop \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} 80\% \quad .$$

- **Ranking and factor investing:**
  - Ranking based on various factors (Value, Low Size, Low Volatility, Momentum, Quality).
  - Factors derived from characteristics like book-to-price ratio, market capitalization, standard deviation, past returns, return on equity.
  - Empirical evidence of excess returns for these factors.

- **Public data ranking methods:**
  - Expected performance:

  $$\text{score} = \boldsymbol{\mu};$$

  - Sharpe ratios:

  $$\text{score} = \frac{\boldsymbol{\mu}}{\sqrt{\text{diag}\left(\boldsymbol{\Sigma}\right)}};$$

  - Mean-variance ratios:

  $$\text{score} = \frac{\boldsymbol{\mu}}{\text{diag}\left(\boldsymbol{\Sigma}\right)}.$$

Heuristic portfolios as robust portfolios

- **The $1/N$ and quintile portfolios as robust portfolios:**
  - In principle, they are just heuristic portfolios.
  - Nevertheless, they can be formally derived as portfolios robust to estimation errors in $\boldsymbol{\mu}$.
- **Robust formulation of GMRP:**
  - Maximizes the worst-case return considering uncertainty in $\boldsymbol{\mu}$:

$$\begin{array}{ll}
\underset{\boldsymbol{w}}{\text{maximize}} & \min_{\boldsymbol{\mu} \in \mathcal{S}} \boldsymbol{w}^{\mathsf{T}} \boldsymbol{\mu} \\
\text{subject to} & \mathbf{1}^{\mathsf{T}} \boldsymbol{w} = 1, \quad \boldsymbol{w} \geq \mathbf{0},
\end{array}$$

  - Uncertainty set $\mathcal{S}$:

$$\mathcal{S} = \{\hat{\boldsymbol{\mu}} + \boldsymbol{e} \mid \|\boldsymbol{e}\|_1 \leq \epsilon\},$$

  where ϵ is the size of the uncertainty region.
- **Connection to quintile and $1/N$ portfolios:**
  - Quintile portfolio emerges as a solution for a specific ϵ.
  - $1/N$ portfolio arises as a solution for a sufficiently large ϵ.
  - Explains the practical performance of these portfolios due to their robustness.
  - Further details in robust portfolios chapter: (Palomar 2024, chap. 14).
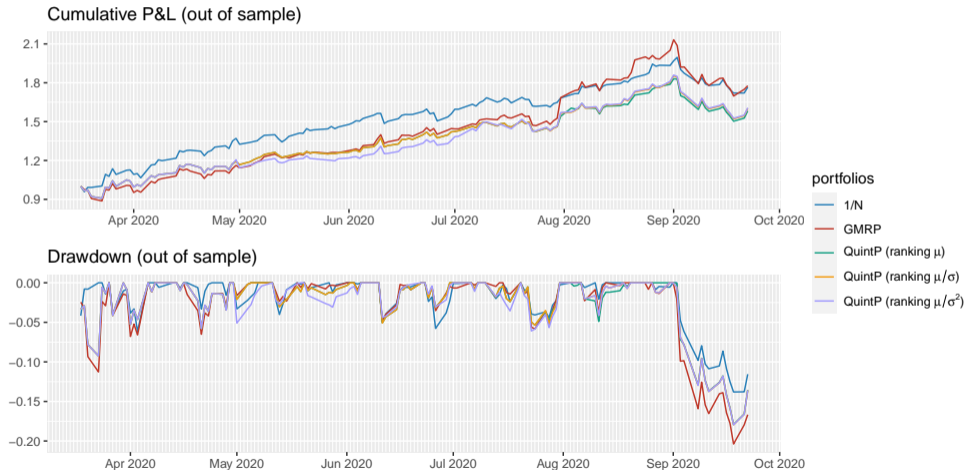
- **Comparison of heuristic portfolios:**
  - Portfolios: GMRP, $1/N$, and quintile portfolios (ranked by μ, μ/σ, μ/σ^2).
  - Data: Eight S&P 500 stocks, 2019-2020.
  - Backtest: 70% training data, 30% test data.

Capital weight allocation for different portfolios
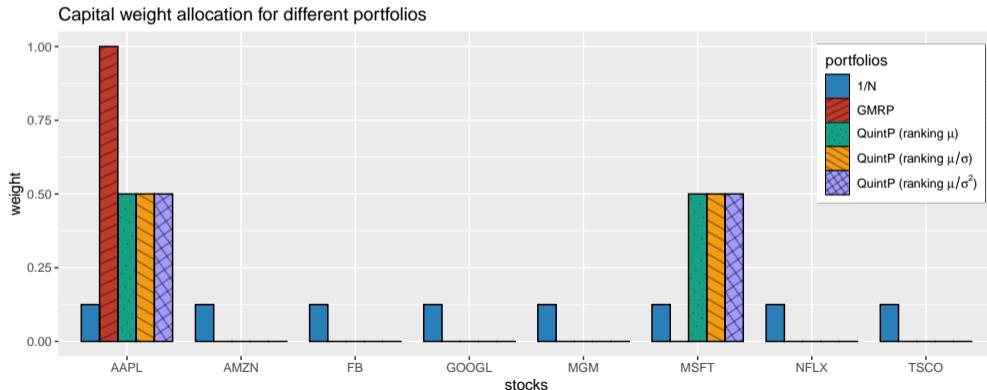
Backtest cumulative P&L and drawdown:

- **Backtest performance analysis:**
  - GMRP shows high returns but increased volatility and drawdown.
  - $1/N$ portfolio exhibits the best drawdown performance, indicating good diversification.

- **Considerations for realistic backtesting:**
  - Multiple rolling-window backtests recommended for robust assessment.
  - Chapter on backtesting provides detailed methodologies.

Outline

Risk-based portfolios

- **Risk-based portfolios overview:**
  - Focus on risk considerations, utilizing the covariance matrix Σ.
  - Ignore return forecasts μ due to their unreliable estimation.
  - Aim to achieve diversification and risk management without relying on expected returns.

- **Characteristics:**
  - Based on the premise that risk parameters (Σ) are more reliably estimated than returns (μ).
  - Seek to optimize portfolio construction through risk minimization or distribution.

- **Significance:**
  - Addresses the challenge of high uncertainty in return forecasts.
  - Provides a framework for portfolio construction when return predictions are deemed too speculative or volatile.

Risk-based portfolios

- **Examples:**
  - Global minimum variance portfolio (GMVP)
  - Inverse volatility portfolio (IVolP)
  - Inverse variance portfolio (IVarP)
  - Risk parity portfolio (RPP)
  - Most diversified portfolio (MDP)
  - Maximum decorrelation portfolio (MDCP)

- **References:**
  - (Ardia et al. 2017) for detailed exploration of risk-based portfolios.
  - (Chopra and Ziemba 1993) highlighting the issues with estimating μ in practice.
  - (Palomar 2024, chap. 6) for a detailed coverage of these slides.

- **Global minimum variance portfolio (GMVP):**
  - Objective: Minimize portfolio variance.
  - Formulation:

  $$\underset{\boldsymbol{w}}{\text{minimize}} \quad \boldsymbol{w}^{\mathsf{T}} \boldsymbol{\Sigma} \boldsymbol{w}$$
  $$\text{subject to} \quad \mathbf{1}^{\mathsf{T}} \boldsymbol{w} = 1, \quad \boldsymbol{w} \geq \mathbf{0}.$$

  - Convex quadratic program (QP) solvable with QP solvers.
- Without shorting constraints it has the closed-form solution:

$$\boldsymbol{w} = \frac{1}{\mathbf{1}^{\mathsf{T}} \boldsymbol{\Sigma}^{-1} \mathbf{1}} \boldsymbol{\Sigma}^{-1} \mathbf{1}.$$

- **Significance:**
  - Utilized for its simplicity and as a benchmark in academic research.
  - Focuses on risk minimization through variance reduction.
  - Often used to evaluate covariance matrix estimators, excluding $\boldsymbol{\mu}$ estimation effects.

Risk-based portfolios: Inverse volatility portfolio (IVolP)

- **Inverse volatility portfolio (IVolP):**
  - Aims to control risk by inversely weighting assets based on their volatility.
  - Defined as:

  $$\boldsymbol{w} = \frac{\boldsymbol{\sigma}^{-1}}{\mathbf{1}^{\mathsf{T}}\boldsymbol{\sigma}^{-1}},$$

  where $\boldsymbol{\sigma}^2 = \text{diag}\left(\boldsymbol{\Sigma}\right)$ and $\boldsymbol{\sigma}$ are the assets' volatilities.

- **Characteristics:**
  - Lower weight to high volatility assets, higher weight to low volatility assets.
  - Also known as *equal volatility portfolio*.
  - Ensures weighted constituent assets have equal volatility contribution to the portfolio.

- **Comparison with GMVP:**
  - The GMVP with $\boldsymbol{\Sigma}$ diagonal leads to the inverse variance portfolio (IVarP) (resembling the IVolP):

  $$\boldsymbol{w} = \frac{\boldsymbol{\sigma}^{-2}}{\mathbf{1}^{\mathsf{T}}\boldsymbol{\sigma}^{-2}},$$

  - Both strategies allocate inversely based on volatility, but the IVarP squares the volatilities (variance).

- **Risk parity portfolio (RPP) or equal risk portfolio (ERP):**
  - Aims to equalize risk contributions from all assets to the total portfolio risk.
  - Considers both asset volatilities and correlations.
  - Detailed exploration in Chapter @ref(RPP).
- **Comparison with IVolP:**
  - IVolP focuses on inversely weighting assets based on their volatilities, ignoring correlations.
  - RPP advances this concept by incorporating asset correlations, offering a more comprehensive risk management approach.
- **Objective:**
  - Distribute risk evenly across assets, unlike $1/N$ portfolio which equalizes capital allocation.
  - Ensures no single asset disproportionately affects the portfolio's overall risk.
- **Significance:**
  - Recognizes and addresses the impact of asset correlations on portfolio risk.
  - Provides a balanced approach to risk management, suitable for diversified investment strategies.

From dollar diversification ($1/N$ portfolio) to risk diversification (RPP):

Portfolio weights

Relative risk contribution
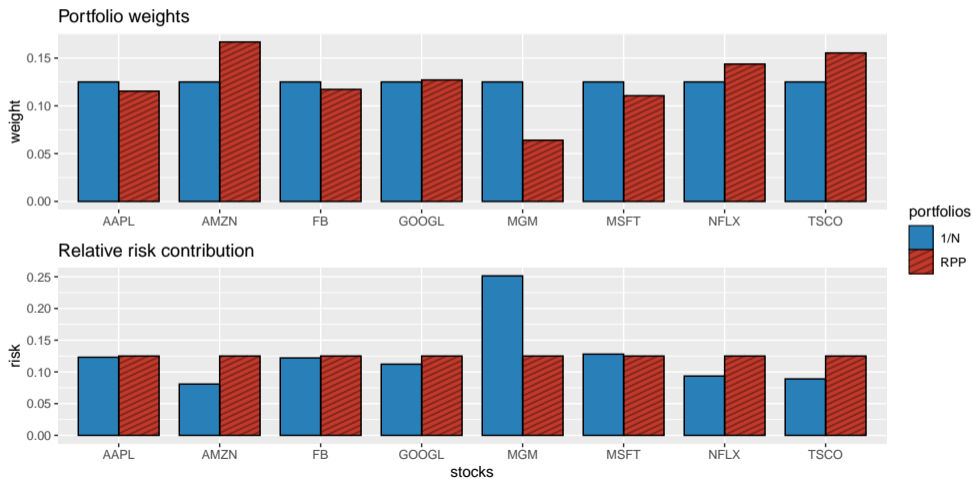
Risk-based portfolios: Most diversified portfolio (MDP)[*]

- **Most diversified portfolio (MDP):**
  - Utilizes the diversification ratio (DR) as a proxy for risk efficiency.
  - Defined as:
  $$DR = \frac{\mathbf{w}^\mathsf{T} \boldsymbol{\sigma}}{\sqrt{\mathbf{w}^\mathsf{T} \boldsymbol{\Sigma} \mathbf{w}}}.$$
  - Long-only portfolios have DR ≥ 1.
- **MDP formulation:** Maximizes DR

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \dfrac{\mathbf{w}^\mathsf{T} \boldsymbol{\sigma}}{\sqrt{\mathbf{w}^\mathsf{T} \boldsymbol{\Sigma} \mathbf{w}}} \\ \text{subject to} & \mathbf{1}^\mathsf{T} \mathbf{w} = 1, \quad \mathbf{w} \geq \mathbf{0}. \end{array}$$

- **Significance:**
  - MDP aims to maximize diversification benefits by considering asset volatilities and correlations.
  - Offers a portfolio construction approach that aligns with the concept of risk efficiency in markets.

- **Maximum decorrelation portfolio (MDCP):**
  - Minimizes the portfolio correlation:

$$\begin{array}{ll} \underset{w}{\text{minimize}} & w^{\mathsf{T}} C w \\ \text{subject to} & \mathbf{1}^{\mathsf{T}} w = 1, \quad w \geq \mathbf{0}. \end{array}$$

  - Correlation matrix C represents covariance of normalized assets with unit volatility.
- **Relation to GMVP and MDP:**
  - MDCP is akin to GMVP with correlation matrix instead of covariance matrix.
  - MDCP weights, when adjusted for asset volatilities, yield MDP weights.
- **Properties of MDCP:**
  - Maximizes the Diversification Ratio (DR) when assets have equal volatility.
  - Maximizes the Sharpe Ratio (SR) when assets have equal risks and returns.
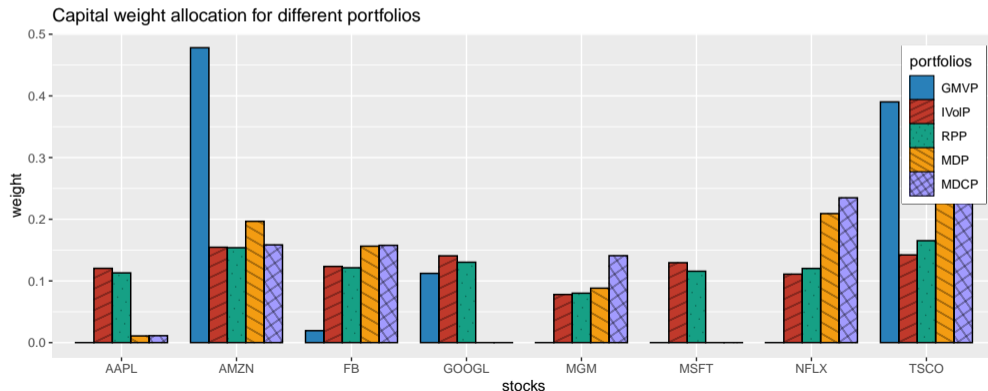- **Significance:**
  - MDCP focuses on reducing portfolio correlation, enhancing diversification.
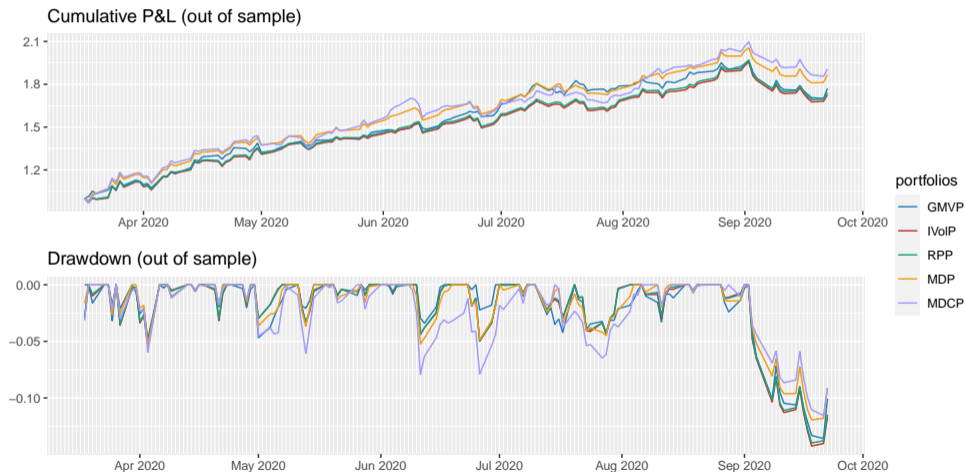 - Offers an alternative risk management strategy by considering asset correlations.

- **Comparison of risk-based portfolios:**
  - Portfolios: GMVP, IVolP, RPP, MDP, and MDCP.
  - Data: Eight S&P 500 stocks, 2019-2020.

Capital weight allocation for different portfolios

Backtest cumulative P&L and drawdown:

Cumulative P&L (out of sample)

Drawdown (out of sample)

- **Backtest performance analysis:**
  - Emphasizes the effectiveness of risk-based portfolios in minimizing drawdowns compared to heuristic portfolios.
  - Showcases the impact of different risk minimization strategies on portfolio performance.

- **Observations:**
  - Risk-based portfolios generally exhibit smaller drawdowns, highlighting their focus on risk minimization.
  - The allocation and performance of these portfolios demonstrate the practical implications of different risk management approaches in portfolio construction.

Outline

Summary

- A portfolio of N assets is represented by vector w, with return $w^\mathsf{T} r$ where r is the asset returns vector.
- Portfolios require periodic rebalancing, which incurs transaction costs that reduce returns.
- Portfolios must meet various constraints, including regulatory, broker-imposed, and investor-specific ones, most of which are convex for easy optimization, except sparsity control.
- Portfolio performance is assessed using metrics like Sharpe ratio and CVaR, and incorporating these into optimization creates complex formulations.
- Simple heuristic portfolios like the $1/N$ and quintile portfolios are easy to compute and historically effective.
- Risk-based portfolios aim to minimize return variability without market trend speculation, offering ease of computation and practical robustness.

Ardia, D., G. Bolliger, K. Boudt, and J.-P. Gagnon-Fleury. 2017. "The Impact of Covariance Misspecification in Risk-Based Portfolios." *Annals of Operations Research* 254 (1-2): 2–16.

Bacon, Carl. 2008. *Practical Portfolio Performance Measurement and Attribution*. 2nd ed. Wiley.

Chopra, V., and W. Ziemba. 1993. "The Effect of Errors in Means, Variances and Covariances on Optimal Portfolio Choice." *Journal of Portfolio Management* 19 (2): 6–11.

Palomar, D. P. 2024. *Portfolio Optimization: Theory and Application*. Cambridge University Press.